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Entanglement in bipartite and tripartite quantum systems

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Chapter 1

Introduction and Objectives

The counterintuitive properties of entanglement were first discussed by Albert Einstein in 1935, in a joint paper with Boris Podolsky and Nathan Rosen [1]. It was Erwin Schrödinger, who shortly thereafter coined the word entanglement and described it as "not one but rather the characteristic trait of quantum mechanics". Although these first studies criticized quantum mechanics, by arguing that quantum description of physical reality is not complete, repeated experiments have verified that photons, ions and more recently, solid-state systems such us quantum dots or SQUIDs (superconducting quantum interference devices), can show this behavior.

This behavior implies the existence of global states of composite systems which cannot be written as a product of the states of individual subsystems. Schrödinger concluded that "best possible knowledge of a whole does not include best possible knowledge of its parts and this is what keeps coming back to haunt us".

In 1964 John Bell accepted the incompleteness conclusion as a working hypothesis and proposed the local hidden variable model (LHVM) [2], summarizing the deterministic world idea by the following assumptions [3]: (i) measurement results are determined by properties the particles carry prior to and independent of the measurement ("realism"), (ii) results obtained at one location are independent of any actions performed at space-like separation ("locality") and (iii) the setting of local apparatus are independent of the hidden variables which determine the local results ("free will"). He then proved that these assumptions impose constraints, in the form of inequalities, on correlations in experiments. The outcomes obtained when suitably measuring some entangled quantum state violate the Bell inequalities.

But it was not until the 80s that a convincing test of violation of the Bell inequalities was performed. However the experiments carried out by Aspect *et al.* [4] and [5] and many others since, still suffer from locality and/or detection loopholes. The ones of the first kind are due to the fact that the two detections are separated by a time-like interval, which means that the first

detection might influence the second one by some kind of signal, whereas the detection loopholes are consequence of particles not always being detected at both wings of the detector. Advancements in technology during the last 30 years have led to significant elimination of loopholes as well as a vast variety of methods to test the Bell inequalities. Therefore, for most physicist, it is by this time unrealistic to hold to a local realistic view.

A main problem is, however, that it is not easy to give a definition of entanglement other than it is a property of entangled states. That is why the problem has been shifted: nowadays the fundamental question in quantum entanglement theory is which states are entangled and which are not [6]. Being a property of correlations between quantum systems, entanglement defies a classical description and, unfortunately, the structure that nature conceals seems to be, in general, very complex.

Other than entanglement detection, much efforts have been devoted to the concept of entanglement manipulation. As well as the interest in testing quantum phenomena, with the ability of manipulating individual quantum systems, the possibility of using quantum correlations as a resource to perform tasks that are inefficient or even impossible by classical means, has become stronger. The development of this concept is a central element in modern quantum information science.

This work is essentially a review of the entanglement phenomenon in bipartite and tripartite quantum systems. However, original calculations both analytically and numerically have been presented: in subsections 3.3.1 the evolution of entanglement under a Ising Hamiltonian is considered; subsection 3.3.2 shows that the PPT criterion identifies entanglement of every pure $3 \otimes 3$ quantum state, and finally in section 4.5 we present an ingenious method to verify whether GHZ and W states belong to the same SLOCC class or not.

Our approach is going to be that of defining entangled states, but before, we shall define states that are not entangled, which is actually simpler. We shall see later on that manipulation of entanglement plays in fact a fundamental role in entanglement theory.

Chapter 2

Mathematical background

In this section we are going to introduce some mathematical concepts that are of great importance for the subject we are treating here.

2.1 Density operator

The state of a quantum system may not be completely known. Mathematically, a quantum state is represented by a state vector $|\psi\rangle$ in a Hilbert space, which we call a pure state. A mixed quantum state corresponds to a probabilistic mixture of pure states and it applies when there is not enough information to specify the normalized state $|\psi\rangle$. If p_n are the probabilities that the system is in a normalized state $|\psi_n\rangle$ then the expectation value of an operator A is

$$\langle A\rangle = \sum_n p_n \langle\psi_n|A|\psi_n\rangle. \tag{2.1}$$

In this context of lack of information, it appears to be convenient to introduce the density operator ρ which is the hermitian operator

$$\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n|. \tag{2.2}$$

Density operators of this form represent a statistical mixture of states. When $\rho = |\psi_i\rangle\langle\psi_i|$ we have a pure state and $\rho^2 = \rho$. Given that $\mathrm{tr}(\rho) = 1$, it follows that $\mathrm{tr}(\rho^2) = 1$ for pure states. For mixed states on the other hand $\mathrm{tr}(\rho^2) = \sum_n p_n^2 < 1$.

Density operators are also positive, which means that for any state $|\phi\rangle$

$$\langle\phi|\rho|\phi\rangle = \sum_n p_n |\langle\phi|\psi_n\rangle|^2 \geq 0, \tag{2.3}$$

because every term in the sum is positive or zero.

It is also convenient to introduce a measure of the uncertainty of the state vector in the form of entropy, the von Neumann entropy

$$S(\rho) = -\mathrm{tr}(\rho \log \rho), \tag{2.4}$$

which multiplying it by Boltzmann's constant gives the thermodynamic entropy.

In 1930, Paul Dirac introduced the idea of the reduced density matrix. In the quantum description of physics, systems are composed by more elementary subsystems. If the system is composed of two subsystems A and B then the possible states of the total system form the total space that is given by the tensor product

$$\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B, \tag{2.5}$$

of Hilbert spaces.

The term subsystem has a large meaning here. It may refer to different particles or to degrees of freedom such as the spin. Let us consider as an example two particles of spin 1/2. $\mathcal{H}_{tot} = \mathbb{C}_1^2 \otimes \mathbb{C}_2^2$ if only the spins are taken into account. The state of the total system can be:

$$|\psi_1\rangle = |+\rangle \otimes |+\rangle. \tag{2.6}$$

Moving to the density matrix formalism this reads:

$$\rho = |\psi\rangle\langle\psi| = |++\rangle\langle++| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.7}$$

in the $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$ basis representation. The reduced density matrices read now as follows:

$$\rho_a = \mathrm{tr}_b(|++\rangle\langle++|) = |+\rangle\langle+|(\langle+|+\rangle)^2 = |+\rangle\langle+| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.8}$$

and equally

$$\rho_b = |+\rangle\langle+| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.9}$$

More generally,

$$\rho_a = \mathrm{tr}_b(\rho) = \sum_n {}_b\langle\psi_n|\rho|\psi_n\rangle_b, \tag{2.10}$$

where $\{|\psi_n\rangle_b\}$ forms the basis that spans the second subsystem and similarly for ρ_b.

2.2 Unitary operators

An operator U is said to be unitary if $U^\dagger U = UU^\dagger = \mathbb{I}$. Unitary operators preserve the inner products between vectors:

$$(U|v\rangle, U|w\rangle) = \langle v|U^\dagger U|w\rangle = \langle v|w\rangle. \tag{2.11}$$

Defining $|w_i\rangle \equiv U|v_i\rangle$, where $|v_i\rangle$ is an orthonormal basis set and therefore so is $|w_i\rangle$, we can write $U = \sum_i |w_i\rangle\langle v_i|$.

Unitary operators are very useful in quantum mechanics because they are generated by Hermitian operators as we are going to prove next. First note that any operator can be written in the form $A + iB$, where A and B are hermitian, and therefore iB is skew-Hermitian, so that

$$U = A + iB, \tag{2.12}$$

$$U^\dagger = A - iB \tag{2.13}$$

and then the unitarity implies that

$$U^\dagger U = A^2 + B^2 + i[A, B] = \mathbb{I}, \tag{2.14}$$

$$UU^\dagger = A^2 + B^2 - i[A, B] = \mathbb{I}. \tag{2.15}$$

It follows now that A and B must commute and then $A^2 + B^2 = \mathbb{I}$, which allows us to write $A = \cos C$ and $B = \sin C$, so that $U = \exp(iC)$.

In particular, the unitary time evolution operator, $U(t) = \exp(-iHt/\hbar)$, is generated by the Hamiltonian H. Considering a initial density operator written in the diagonalized form

$$\rho(0) = \sum_n p_n |\psi_n(0)\rangle\langle\psi_n(0)|, \tag{2.16}$$

the formal solution of the Schrödinger equation is $|\psi_n(t)\rangle = U(t)|\psi_n(0)\rangle$ and along with the bra equivalent we are able to write

$$\rho(t) = \sum_n p_n U(t)|\psi_n(0)\rangle\langle\psi_n(0)|U^\dagger(t) = U(t)\rho(0)U^\dagger(t). \tag{2.17}$$

It is a unitary evolution of ρ.

2.3 Von Neumann measurements

Von Neumann or projective measurements are the kind of measurements that are introduced in introductory quantum theory courses within the scheme of the postulates of quantum mechanics. Here this process is going to be defined using the density operator formalism.

If a quantum system is described by a density operator ρ and if $A = \sum \lambda_n |\lambda_n\rangle\langle\lambda_n|$ is an operator associated to an observable, then after the measurement of the magnitude A, the probability of getting the result λ_n is

$$p(\lambda_n) = \langle\lambda_n|\rho|\lambda_n\rangle = \mathrm{tr}(\rho|\lambda_n\rangle\langle\lambda_n|) = \mathrm{tr}(\rho P_n). \tag{2.18}$$

Here the operator P_n is called the projector over the space of the eigenvalue λ_n of A. A von Neumann measurement is one in which the probability for a given measure is given in this form. If the eigenstates of A are degenerate, then the probability for the corresponding eigenvalue will read

$$p(\lambda_n) = \sum_i \mathrm{tr}(\rho|\lambda_n^i\rangle\langle\lambda_n^i|). \tag{2.19}$$

Now, the reduction of the wave-packet postulate tells us that the measurement is accompanied by a change in the density operator of the form

$$\rho \to \rho_n' = \frac{P_n \rho P_n}{\mathrm{tr}(P_n \rho P_n)} = \frac{P_n \rho P_n}{p(\lambda_n)}, \tag{2.20}$$

that is, the projection of ρ onto the space of eigenstates associated with the measurement result divided by the probability for the observed measurement outcome, so that normalization is ensured. The last step follows from the cyclicity of the trace, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, and from the fact that projectors are orthonormal, which means that $P_i P_j = P_i \delta_{ij}$.

2.4 Qubit

A qubit is a quantum system having two orthogonal states. It is the quantum analogue to the term bit in information theory and it can hold one bit by preparing it either in the state $|0\rangle$ or $|1\rangle$. However, due to the superposition principle in quantum mechanics, a qubit can be also prepared in any superposition state of the form

$$|Qubit\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{2.21}$$

where α and β are complex numbers.

A physical implementation of a qubit can be provided in principle by any quantum system with two states such as the two orthogonal polarization states of a photon or the orientation of a spin-half particle, associating conventionally $|+\rangle$ to $|0\rangle$ and $|-\rangle$ to $|1\rangle$.

It is helpful to represent the qubit states as points on the surface of a sphere, the Bloch sphere depicted in figure 2.1. North and south poles correspond respectively to $|0\rangle$ and $|1\rangle$ and more generally, opposite points represent mutually orthogonal states. Finally the x and y axes hold the eingenstates of σ_x and σ_y respectively.

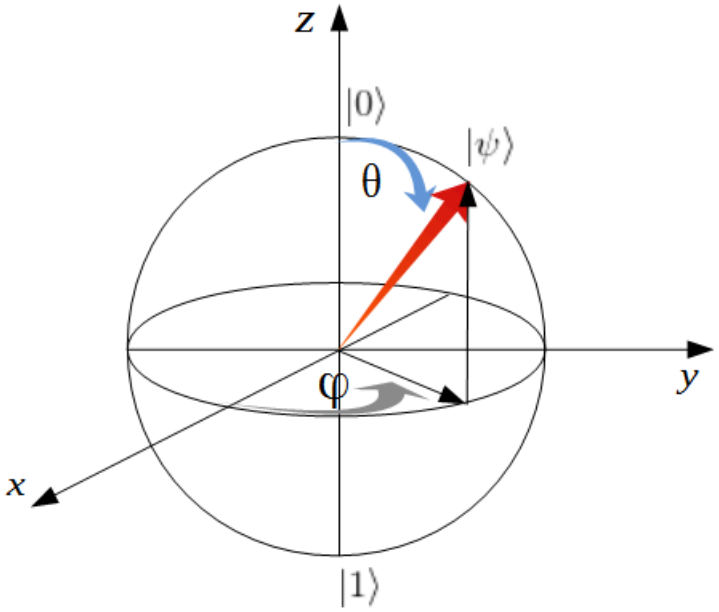

Figure 2.1: Bloch sphere.

Other than the generic qubit pure state

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle, \tag{2.22}$$

mixed states can also be visualized in this representation. Thanks to its hermiticity, the density operator ρ can be written as a combination of the four Pauli operators

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.23}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.24}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{2.25}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.26}$$

with real coefficients a, b and c:

$$\rho = \frac{1}{2}(\mathbb{I} + a\sigma_x + b\sigma_y + c\sigma_z). \tag{2.27}$$

This permits as to associate a, b and c with the x, y and z components of the Bloch vector and the eigenvectors $|\lambda\rangle$ and $|\phi\rangle$ of ρ are also eigenvectors of $a\sigma_x + b\sigma_y + c\sigma_z$ corresponding to the eigenvalues $\pm\sqrt{a^2+b^2+c^2}$. Therefore, we can write the diagonalized density operator ρ as follows:

$$\rho = \frac{1}{2}(1+\sqrt{a^2+b^2+c^2})|\lambda\rangle\langle\lambda| + \frac{1}{2}(1-\sqrt{a^2+b^2+c^2})|\phi\rangle\langle\phi|. \tag{2.28}$$

This reduces to the pure state $|\lambda\rangle\langle\lambda|$ for $a^2+b^2+c^2 = 1$, that is, for the states lying on the surface of the Bloch sphere. For $a^2+b^2+c^2 < 1$, on the other hand, we get mixed states and the Bloch vector describes a point inside the sphere.

Furthermore, unitary operators on single qubits are naturally visualized in the Bloch representation. One way of seeing this is writing them as a composition of the rotation operators about the x, y and z axes

$$R_x(\epsilon) = \exp\left(-i\frac{\epsilon}{2}\sigma_x\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-i\frac{\epsilon}{2}\sigma_x\right)^n = \cos\left(\frac{\epsilon}{2}\right)\mathbb{I} - i\sin\left(\frac{\epsilon}{2}\right)\sigma_x, \tag{2.29}$$

$$R_y(\epsilon) = \exp\left(-i\frac{\epsilon}{2}\sigma_y\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-i\frac{\epsilon}{2}\sigma_y\right)^n = \cos\left(\frac{\epsilon}{2}\right)\mathbb{I} - i\sin\left(\frac{\epsilon}{2}\right)\sigma_y, \tag{2.30}$$

$$R_z(\epsilon) = \exp\left(-i\frac{\epsilon}{2}\sigma_z\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-i\frac{\epsilon}{2}\sigma_z\right)^n = \cos\left(\frac{\epsilon}{2}\right)\mathbb{I} - i\sin\left(\frac{\epsilon}{2}\right)\sigma_z. \tag{2.31}$$

These operators are themselves unitary and so is

$$\begin{aligned} U =& e^{i\chi}R_z(\delta)R_y(\mu)R_z(\nu) \\ =& \begin{pmatrix} e^{i(\chi-\delta/2-\nu/2)}\cos\frac{\mu}{2} & -e^{i(\chi-\delta/2+\nu/2)}\sin\frac{\mu}{2} \\ e^{i(\chi+\delta/2-\nu/2)}\sin\frac{\mu}{2} & e^{i(\chi+\delta/2+\nu/2)}\cos\frac{\mu}{2} \end{pmatrix}. \end{aligned} \tag{2.32}$$

Here, δ, μ and ν are the Euler angles that describe an orientation in 3-dimensional euclidean space and χ simply acts to change the global arbitrary phase of the state vector. By combining such real numbers we can construct any unitary operation on single qubits.

Chapter 3

Bipartite entanglement

3.1 EPR paradox

The EPR paradox [1], named after Albert Einstein, Boris Podolsky and Nathan Rosen, was a thought experiment which revealed what later would be called entanglement. Bohm presented the EPR paradox in terms of a pair of spin 1/2 particles prepared in a zero total angular momentum state [7]:

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|+\rangle_A|-\rangle_B - |-\rangle_A|+\rangle_B). \tag{3.1}$$

Two distant parties, called usually Alice and Bob in computer and quantum information sciences, each have one of the pair of the quantum states, A and B respectively. Translating it now to the qubit notation this reads

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B). \tag{3.2}$$

If Alice chooses to measure σ_z then she immediately establishes that the state of Bob's qubit is $|1\rangle_B$ or $|0\rangle_B$, corresponding, respectively, to her results +1 and -1. If Alice measures σ_x, however, then what she establishes is that the state of Bob's qubit is $|0'\rangle_B = 2^{-1/2}(|0\rangle_B + |1\rangle_B)$ or $|1'\rangle_B = 2^{-1/2}(|0\rangle_B - |1\rangle_B)$, corresponding again to her results -1 and +1 respectively, as

$$\begin{aligned} |\psi\rangle_{AB} =& \frac{1}{2\sqrt{2}}((|0'\rangle_A + |1'\rangle_A) \otimes (|0'\rangle_B - |1'\rangle_B) \\ & - (|0'\rangle_A - |1'\rangle_A) \otimes (|0'\rangle_B + |1'\rangle_B)) \\ =& \frac{1}{\sqrt{2}}(|1'\rangle_A|0'\rangle_B - |0'\rangle_A|1'\rangle_B). \end{aligned} \tag{3.3}$$

σ_z and σ_x have no common eigenstates and therefore there is no quantum state having well-defined values for both observables. Then by the choice

of measuring one or the other, Alice establishes either of two incompatible properties of Bob's qubit that should not be possible to establish at the source where the qubits were created. Alice's measure seems to change instantaneously Bob's qubit and this conflicts with what is called local realism. According to locality, physical influences should not propagate from one party to the other at a speed greater than that of light. Realism lies on the belief that the properties of Bob's qubit exist prior to and independently of the measurement.

3.2 Bell states and Schmidt decomposition

The Bell states are usually presented as the most simple examples of entangled states. Furthermore they have been realized in a number of diverse experiments. They are conventionally written in the following form:

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle), \tag{3.4}$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle), \tag{3.5}$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle), \tag{3.6}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle - |1\rangle|1\rangle). \tag{3.7}$$

They are also known as the maximally entangled two-qubit states. Let us write for example the antisymmetric state used in the Bohm's version of the EPR experiment:

$$\rho = |\Psi^-\rangle\langle\Psi^-| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.8}$$

Evaluating the reduced density operator either for the a or the b subsystem we get

$$\rho_a = \mathrm{tr}_b(\rho) = \frac{1}{2}(|0\rangle\langle 0|(\langle 1|1\rangle)^2 + |1\rangle\langle 1|(\langle 0|0\rangle)^2) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \rho_b. \tag{3.9}$$

So the reduced density matrix corresponds to a mixed state. This is somewhat surprising as the total system is pure. It means that it is not possible to specify the exact state of a single qubit when this is entangled to another one. In other words, our entangled state is not separable. Thus, $\mathrm{tr}(\rho^2) < 1$

for a subsystem of a bipartite pure state is a signature of entanglement. Quantum superposition leads to a kind of correlations that cannot be explained by classical means and it is by the word entanglement that this phenomena is known.

Before giving a more precise definition of entanglement, which in fact it is going to be that of separability, let us define the Schmidt decomposition [8]. It is always possible to write an entangled pure state as summation of the orthonormal sets $|\lambda_n\rangle$ and $|\phi_n\rangle$, by a suitable choice of the local unitary operators U_A and U_B:

$$|\psi\rangle = U_A \otimes U_B \sum_{ij} a_{ij}|i\rangle_a|j\rangle_b = \sum_n a_n|\lambda_n\rangle_a|\phi_n\rangle_b, \tag{3.10}$$

where a_n are non-negative real numbers and a_{ij} the elements of a diagonalizable matrix $A \to U_A A U_B^T$ that completely represents the state provided that the basis has been specified. This form is known as the Schmidt decomposition and the orthonormal states are the eigenstates of the reduced density operators

$$\rho_a = \sum_n a_n^2 |\lambda_n\rangle\langle\lambda_n|, \tag{3.11}$$

$$\rho_b = \sum_n a_n^2 |\phi_n\rangle\langle\phi_n|. \tag{3.12}$$

Both density operators have the same eigenvalues a_n^2 and if the states $|\lambda_n\rangle$ and $|\phi_n\rangle$ are the eigenstates of a pair of operators

$$A = \sum_n \lambda_n |\lambda_n\rangle\langle\lambda_n|, \tag{3.13}$$

$$B = \sum_n \phi_n |\phi_n\rangle\langle\phi_n| \tag{3.14}$$

and if the eigenvalues are distinct then it follows that the outcome of a measurement of B is uniquely determined by a measurement of A and vice versa. Observables A and B are perfectly correlated.

Pure states that are not entangled have a corresponding Schmidt decomposition which has one and only one Schmidt coefficient.

3.3 Mixed entangled states and PPT criterion

The existence of correlated properties, of course, is not something particular to entangled states. Indeed, unlike pure states, all correlated mixed states are not entangled. Let us first introduce an uncorrelated mixed state of two systems A and B defined on $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$:

$$\rho = \rho_A \otimes \rho_B. \tag{3.15}$$

A correlated but not entangled state will be one formed by a mixture of states of these kind and it will not display the intrinsically quantum correlations associated with entanglement. It is a separable state:

$$\rho = \sum_i p_i \, \rho_A^i \otimes \rho_B^i. \tag{3.16}$$

An entangled mixed state is the one that cannot be represented nor approximated by this form [6].

It would be useful to have a universal method to tell whether a given state ρ is entangled or not but in general the problem of separability of mixed states appears to be extremely complex and finding one such method is still an important research problem in quantum information theory. There are however some operational criteria for some cases.

For $2 \otimes 2$ and $2 \otimes 3$ cases there is a sufficient and necessary condition for a mixed state to be entangled known as the positive partial transpose (PPT) criterion [9]. Let us write a given state in the following form:

$$\rho = \sum_{mn} \rho_{mn} \, |m\rangle\langle n|, \tag{3.17}$$

for a single qubit the desity matrix will be:

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{01}^* & \rho_{11} \end{pmatrix} \tag{3.18}$$

as ρ is an hermitian operator. The transpose of the density operator reads

$$\rho^T = \begin{pmatrix} \rho_{00} & \rho_{01}^* \\ \rho_{01} & \rho_{11} \end{pmatrix}, \tag{3.19}$$

which itself represents a possible density operator for a quantum system, given that the transpose operation does not change the eigenvalues of a matrix.

The partial transpose operation performs the transpose on one of the subsystems so that an unentangled mixed state will become

$$\rho^{PT_B} = \sum_{ij} p_{ij} \, \rho_A^i \otimes (\rho_B^j)^T, \tag{3.20}$$

if the transpose is operated on subsystem B. This represents an allowed state for the whole system given that so does the substate $(\rho_B^i)^T$ for the B subsystem. The same thing happens if we take the transpose on subsystem A, in fact, it turns out that $\rho^{PT_A} = (\rho^{PT_B})^T$, what permits us to use either of them indistinctively.

Now, if the partial transpose is performed on an entangled state we often (always for $2\otimes 2$ and $2\otimes 3$ entangled cases) find that the ρ^{PT} has one or more negative eigenvalues and as this is not permitted for a density operator, we conclude that the state is indeed an entangled one. In other words, the PPT is necessary but not sufficient to guarantee that a state is separable for states bigger than $2 \otimes 3$.

We have developed, using MATLAB computing language, a function that computes the partial transpose of a matrix and a little program that applies it for the PPT criterion:

```
function y = ptbipartite(x, d1, d2, dppt)
%pttbipartite calculates the partial transpose of a
%bipartite state x
%   d1, d2 and dppt are the dimensions of the 1st and 2nd
%   subsystems and the dimension of the subsystem to take
%   the transpose of respectively.
dnoppt=d1*d2/dppt;
yy=0*x;
y=0*x;
for m=1:dnoppt
    for n=1:dnoppt
        yy=(x((m-1)*dppt+1:dppt*m,(n-1)*dppt+1:dppt*n))';
        y((m-1)*dppt+1:dppt*m,(n-1)*dppt+1:dppt*n)=yy;
    end
end
x
y
end
```

```
aa=input('Enter the matrix representing the bipartite state');
ab=input('Enter the dimension of one subsystem');
ba=input('Enter the dimension of the other subsystem');
bb=input('Enter the dimension of the subsystem to transpose');
a=eig(ptbipartite(aa,ab,ba,bb))
if a>=-0.0000001
    'The state is separable'
else 'The state is entangled'
end
```

For example the antisymmetric $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle)$ Bell state used in the EPR experiment has an associated partial transpose that reads

$$\rho^{PT} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.21}$$

One of the eigenvalues of this operator is negative and hence the state is entangled.

3.3.1 Entanglement under unitary evolution

A more interesting example is to consider the evolution of a state under a certain Hamiltonian and see how the entanglement properties change. We present a one-dimensional Ising model representing two spins-1/2. Each spin is allowed to interact with its neighbor and there is no external field interacting with the lattice. Then, the Hamiltonian reads as follows:

$$H = -J(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+), \tag{3.22}$$

where J characterizes the interaction and

$$\sigma_1^+\sigma_2^- = (|0\rangle\langle 1|) \otimes (|1\rangle\langle 0|) = |01\rangle\langle 10| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.23}$$

$$\sigma_1^-\sigma_2^+ = (|1\rangle\langle 0|) \otimes (|0\rangle\langle 1|) = |10\rangle\langle 01| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.24}$$

The diagonalization of the Hamiltonian leads to

$$H_d = J \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.25}$$

$\{|00\rangle, |\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), |11\rangle\}$ being the eigenvectors. Now that the Hamiltonian is diagonalized, let us look at the time evolution of a state that is not stationary, for example $|01\rangle$:

$$\begin{aligned} U(t)|01\rangle &= \exp(-iHt/\hbar)\frac{1}{\sqrt{2}}(|\Psi^+\rangle + |\Psi^-\rangle) \\ &= \frac{1}{2}((|01\rangle + |10\rangle)\exp(iJt/\hbar) + (|01\rangle - |10\rangle)\exp(-iJt/\hbar)) \\ &= \cos(Jt/\hbar)|01\rangle + i\sin(Jt/\hbar)|10\rangle. \end{aligned} \tag{3.26}$$

We have used our MATLAB function to see how the entanglement evolves. The evolution of the negative eigenvalue of the partially transposed density matrix, along with the von Neumann entropy of the reduced density matrix ρ_A, are depicted in figure 3.1. In figure 3.2, on the other hand, all the eigenvalues have been plotted using the Mathematica piece of software, in order to compare with our results. In both figures $J/\hbar = 1$ applies. When the

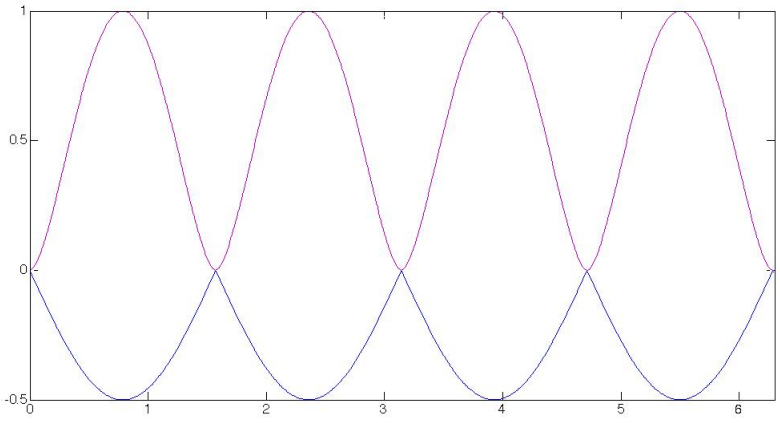

Figure 3.1: numerical calculation of the minimal eigenvalue of the partially transposed matrix and the von Neumann entropy of ρ_A, in blue and violet respectively.

value of the entropy is 1 for the reduced density operator, the total state is a Bell state. This agrees with the reasoning in the previous section of the reduced density matrix being in its most mixed form when the total state is maximally entangled.

It is clear after this example that, unlike what happens under the effect of local unitaries, entanglement can change under global unitary evolution.

3.3.2 PPT for $3 \otimes 3$ pure states

Finally we have also tested states of higher dimension in order to investigate whether, not only the necessity but also, the sufficiency of the PPT criterion holds al least for pure states. We tried with a system made of two qutrits, that is, a $3 \otimes 3$ quantum system. In this case, 3 Schmidt coefficients are enough to write down any entangled pure state, for example:

$$|abc\rangle = a|02\rangle + b|10\rangle + c|21\rangle. \tag{3.27}$$

Then the density matrix has a maximum of 9 positive terms that depend on our three Schmidt coefficients a,b and c:

```
In[1]:= Eigenvalues[{{0, 0, 0, -I * Sin[t] * Cos[t]}, {0, Cos[t]^2, 0, 0},
          {0, 0, Sin[t]^2, 0}, {I * Sin[t] * Cos[t], 0, 0, 0}}]
```

$$\text{Out[1]= } \left\{\text{Cos}[t]^2, -\frac{\sqrt{1-\text{Cos}[4t]}}{2\sqrt{2}}, \frac{\sqrt{1-\text{Cos}[4t]}}{2\sqrt{2}}, \text{Sin}[t]^2\right\}$$

```
Plot[{Cos[t]^2, -Sqrt[1 - Cos[4 t]]/(2 Sqrt[2]), Sqrt[1 - Cos[4 t]]/(2 Sqrt[2]), Sin[t]^2},
 {t, 0, 2 Pi}, PlotLegends → "Expressions"]
```

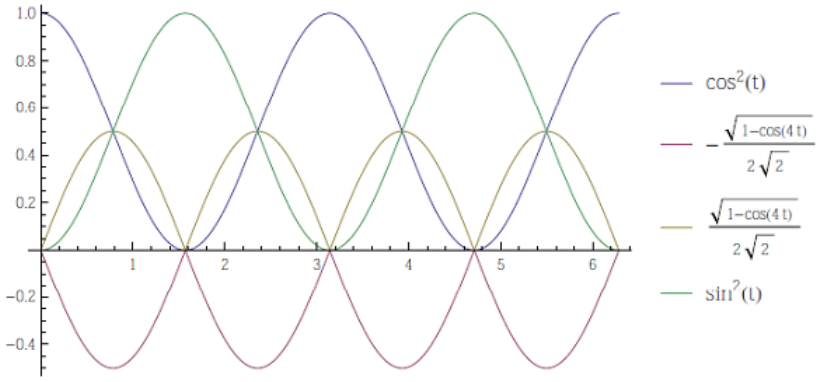

Figure 3.2: evolution of the eigenvalues of the partially transposed density matrix given by Mathematica.

$$|abc\rangle\langle abc| = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & ab & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ac \\ 0 & 0 & a^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & bc & 0 & 0 \\ ab & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & bc & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c^2 & 0 \\ 0 & ac & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.28}$$

Having the 9×9 density matrix parametrized in such way, we calculated the partial transpose and then solved the eigenvalue problem with Mathematica. As much as three eigenvalues turn out to be negative, namely, $-ab, -ac$ and $-bc$. Any of the 5 other possible Schmidt decompositions made of the same three coefficients a, b and c, can be reached using local unitaries from expression 3.27. One simply needs to apply the local identity $U_1 = \mathbb{I}$ for the first qutrit and local unitaries of the form $U_2 = |i\rangle\langle 1| + |j\rangle\langle 2| + |k\rangle\langle 0|$, where $i, j, k \in \{0, 1, 2\}$ and $i \neq j \neq k \neq i$, for the second one.

For the states $|ab\rangle = a|02\rangle + b|10\rangle$, $|ac\rangle = a|02\rangle + c|21\rangle$ and $|bc\rangle = b|10\rangle + c|21\rangle$ we also find a negative eigenvalue in each of the respective partially transposed density matrices.

Therefore, we conclude that every entangled 2-qutrit pure state is identified by the PPT criterion. In fact, a certain degree of mixture is needed in order an entangled state not to be identified by the PPT criterion [10].

Chapter 4

Manipulation and classification of entanglement

4.1 Introduction to different kinds of entanglement

For states composed of more than two subsystems, the variety of entangled states is much richer.

Let us consider first the simplest case in which only two of the three qubits are entangled with each other. One such state could be

$$\begin{aligned} |\Psi\rangle =& \frac{1}{2}(|010\rangle + |100\rangle + |011\rangle + |101\rangle) \\ =& \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle). \end{aligned} \tag{4.1}$$

It is a state that can be separated in a Bell state and in a single qubit and consequently it is said to be partially separable. This state is in fact a biseparable state, a concept that will be introduced more formally in subsequent subsections, where entanglement classes are going to be defined.

However, three qubits can be fully entangled, that is to say, the properties of any qubit are correlated with both of the others. An example of a such state is the Greenberg-Horne-Zeilinger (GHZ) state [11]:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \tag{4.2}$$

We can evaluate the reduced density operator for each subsystem to show that no pure state can be associated to a single qubit in the same way we did for bipartite pure cases. The density operator of the GHZ state takes the following form in the $\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$ basis:

$$|GHZ\rangle\langle GHZ| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.3}$$

Indexing our three subsystems a, b and c, it turns out that we get a mixed state tracing here over the qubit c:

$$\begin{aligned} \rho_{ab} = \mathrm{tr}_c(|GHZ\rangle\langle GHZ|) &= \frac{1}{2}(|00\rangle\langle 00|\langle 0|0\rangle + |11\rangle\langle 11|\langle 1|1\rangle) \\ &= \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{4.4}$$

This remaining mixed state has no entanglement at all. We would have gotten the same result, of course, had we traced out any of the other two qubits.

Tracing over either of the remaining qubits gives the same reduced density matrix we got in the bipartite pure example:

$$\rho_a = \rho_b = \rho_c = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.5}$$

Again, these mixed states tell us that the $|GHZ\rangle$ state is fully entangled.

Another example of an entangled three qubit pure state is the W state [12]

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle), \tag{4.6}$$

which gives

$$|W\rangle\langle W| = \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.7}$$

and

$$\rho_{ab} = \rho_{ac} = \rho_{bc} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.8}$$

so that

$$\rho_a = \rho_b = \rho_c = \frac{1}{3}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.9}$$

An important difference with respect to the GHZ state is the form of the reduced density matrix ρ_{ab}. Taking its partial transpose we find that all eigenvalues of the resulting matrix are not positive or zero and therefore, according to the PPT criterion, ρ_{ab} is entangled. Unlike the entanglement of the GHZ state, the one of the W can be said to be robust when disposing one of the three qubits because the remaining state retains some entanglement.

In the following subsections we are going to introduce entanglement classes and a kind of operations that are necessary in order to understand them.

4.2 LOCC tasks and entanglement classification

Local quantum operations and classical communications (LOCC) [13] is a method where, as the name tells us, operations are performed on part of the total system and where the result of the measure is communicated classically to the other parts. These parts can then perform another local operation on their subsystems, depending on measurements of other parties. Let us illustrate this with an example using the following two Bell states:

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{4.10}$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{4.11}$$

For both states, Alice is in possession of the first qubit while Bob owns the second one, as usual. They can choose measuring one of the states, although they do not know which state $|\Psi^+\rangle$ or $|\Phi^+\rangle$ they are measuring exactly. Assuming that Alice measures σ_z on her qubit and that communicates her result, which is either 1 or -1, to Bob using a classical communications channel, then Bob will measure either -1 or 1 respectively in case they are using the $|\Psi^+\rangle$ state. However, if they are using the $|\Phi^+\rangle$ state, then Bob will measure either 1 or -1 respectively. So after receiving Alice's message with the outcome of her measurement and performing his own measurement, Bob is able to distinguish which of the two states they have been using.

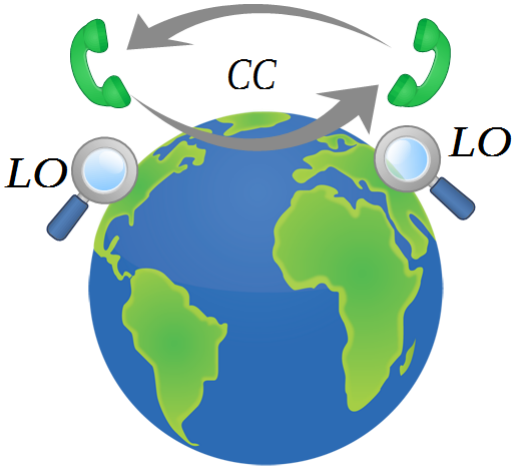

Figure 4.1: Scheme of a possible LOCC arrangement.

But what is the precise meaning of a local operation? Local operations are operations that can be achieved by the composition of the following fundamental steps [14]:

- **Unilocal unitary transformations**. Operations of the form

$$U \otimes \mathbb{I} \otimes ... \otimes \mathbb{I}, \tag{4.12}$$

  that is, the identity for all parties except for one, for which a unitary operator acts. For instance, in a bipartite system this kind of operations are either of the form $U_A \otimes \mathbb{I}_B$ or $\mathbb{I}_A \otimes U_B$.

- **Unilocal von Neumann measurements**. The identity acts for all the parties except for one, on which a von Neumann measurement is carried out:

$$P_n \otimes \mathbb{I} \otimes ... \otimes \mathbb{I}, \tag{4.13}$$

  P_n being the corresponding projector.

- **Addition or subtraction of an uncorrelated ancilla**. One of the parties either couples or decouples an additional, or ancillary, quantum system to its subsystem, so that the density operator of the whole system transforms in the following ways:

$$\rho \to \rho' = \rho \otimes \rho_{anc}, \tag{4.14}$$

$$\rho \to \rho'' = \mathrm{tr}_{anc}(\rho), \tag{4.15}$$

respectively.

Classical communication, on the other hand, means that either zero or one is sent from Alice to Bob and not a superposition of both. This merely allows local operations by one party to be conditioned on the outcome of measurements performed earlier by other parties.

In the spirit of non-local properties so closely related to entanglement, it may appear quite natural to view states that differ only by local operations as equivalently entangled. However LOCC-based classification of entanglement appears to be extremely complicated as we shall see.

Let us first drag our attention to the most intuitive case of separable states. States of the form

$$\rho_{ABC...} = \sum_i p_i \, \rho_A^i \otimes \rho_B^i \otimes \rho_C^i \otimes ... \tag{4.16}$$

of many parties A, B, C, etc. can be created from scratch by means of LOCC. First Alice acts on her subsystem in order to sample from a previously known probability distribution p_i. Now by telling all other parties her outcome, each one acts on his or her party so that every one gets his or her ρ_X^i and then discards the information about the probability of that outcome. As these states satisfy a local hidden variable model (LHVM), their correlations do not defy a classical description. We conclude, as well, that LOCC cannot produce entanglement from an unentangled state.

We are now in position to justify the concept of maximally entangled two-qubit states we introduced as synonym of Bell states. The name comes from the fact that these state are more entangled than all others and, as we shall see now, all other states can be created from the maximally entangled ones by means of LOCC alone.

Limiting ourselves first to only using local unitaries (LU), we can try to characterize bipartite states using the Schmidt normal form

$$|SNF\rangle = \sum_n^{min(d_1,d_2)} \sqrt{p}_n \, |\lambda_n \lambda_n\rangle, \tag{4.17}$$

where d_1 and d_2 are the dimension of the subsystems' Hilbert spaces and p_n are probabilities.

From the definition $U^\dagger U = UU^\dagger = \mathbb{I}$ of unitary operators follows that local unitaries are invertible, that is, a local unitary can be inverted by a local unitary to retrieve the departure state. Hence, and as LOCC cannot produce entanglement, states related by local unitaries have the same amount of entanglement. Therefore, any bipartite quantum state can be written in the Schmidt normal form without affecting the entanglement properties.

Two quantum states are equivalent under local unitary transformations if and only if their normal forms coincide and therefore, it follows that all classes can be parametrized by the angle θ in the two qubits case:

$$\cos\theta\,|00\rangle + \sin\theta\,|11\rangle. \tag{4.18}$$

This is possible thanks to the ability of diagonalizing each of the reduced density matrices, something that is achieved with the adequate local unitary. If $\theta = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}$, then we have a product state whereas when $\cos\theta$ and $\sin\theta$ are equal in magnitude, then the state is going to be most strongly entangled.

We conclude that a continuous parameter such as θ, appears to be necessary in order to label all equivalence classes under local unitaries.

Now let us see how different the situation looks like using LOCC. We will start from the maximally entangled

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{4.19}$$

state and apply the following LOCC recipe, that solves an exercise proposed in [15]:

Alice begins by adding an ancilla in the state $|0\rangle$ that gives

$$\frac{1}{\sqrt{2}}(|00\rangle_A|0\rangle_B + |01\rangle_A|1\rangle_B). \tag{4.20}$$

Then, she applies the local unitary operation that takes $|00\rangle$ to $\cos\theta|00\rangle + \sin\theta|11\rangle$ and $|01\rangle$ to $\sin\theta|01\rangle + \cos\theta|10\rangle$, namely

$$\left(\begin{pmatrix} \cos\theta & 0 & 0 & -\sin\theta \\ 0 & \sin\theta & -\cos\theta & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & 0 & \cos\theta \end{pmatrix} \otimes \mathbb{I}_B\right) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & 0 & 0 & 0 & 0 & 0 & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & 0 & 0 & 0 & 0 & -\sin\theta \\ 0 & 0 & \sin\theta & 0 & -\cos\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin\theta & 0 & -\cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & 0 & \sin\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & 0 & \sin\theta & 0 & 0 \\ \sin\theta & 0 & 0 & 0 & 0 & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & 0 & 0 & 0 & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}(\cos\theta|00\rangle_A|0\rangle_B + \sin\theta|01\rangle_A|1\rangle_B + \cos\theta|10\rangle_A|1\rangle_B + \sin\theta|11\rangle_A|0\rangle_B). \tag{4.21}$$

Separating the part corresponding to the ancilla this reads

$$\frac{1}{\sqrt{2}}(|0\rangle_{anc}(\cos\theta|00\rangle + \sin\theta|11\rangle) + |1\rangle_{anc}(\cos\theta|01\rangle + \sin\theta|10\rangle)). \tag{4.22}$$

Finally Alice performs a projective measurement on the ancilla that yields two possible outcomes that can be communicated classically to Bob. If Alice finds the ancilla in the $|0\rangle$ state then the remaining substate is

$$\frac{1}{\sqrt{2}}(\cos\theta|00\rangle + \sin\theta|11\rangle), \tag{4.23}$$

that is, the generic Schmidt normal form for two qubits.

On the other hand, if Alice finds $|1\rangle$ for the ancilla, then, the result is

$$\frac{1}{\sqrt{2}}(\cos\theta|01\rangle + \sin\theta|10\rangle). \tag{4.24}$$

After being informed of Alice's outcome, the only thing Bob needs to do in order to get the Schmidt normal form is to apply the following local unitary operation:

$$(\mathbb{I}\otimes\sigma_x)\frac{1}{\sqrt{2}}(\cos\theta|01\rangle + \sin\theta|10\rangle)$$

$$= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(\cos\theta|00\rangle + \sin\theta|11\rangle), \tag{4.25}$$

result that could have been obtained just by noting that the effect of σ_x is to convert $|0\rangle$ into $|1\rangle$ and vice versa. So every branch does the job of achieving any two qubit state from the maximally entangled one and given that any mixed state ρ can be written in terms of its eigenvectors $|\psi_i\rangle = ({U_A}^i \otimes {U_B}^i)(\cos\theta_i|00\rangle + \sin\theta_i|11\rangle)$ as

$$\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|, \tag{4.26}$$

then, this is also true for mixed states.

More generally, any bipartite state of n-dimensional subsystems can be prepared with certainty from the maximally entangled

$$|\Phi_n{}^+\rangle = \frac{1}{\sqrt{n}}(|0,0\rangle + |1,1\rangle + ... + |n-1,n-1\rangle) \tag{4.27}$$

state, by means of LOCC alone.

However, it is easy to see that there exist pairs of states that cannot be converted one into the other with certainty. For example, considering a bipartite system formed by two 3-level subsystems or qutrits, there is total certainty of success for the transformation

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \to \frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}|11\rangle, \tag{4.28}$$

using a certain LOCC protocol.

Now for a system formed by a pair of qutrits we can also consider the following transformation:

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \to \frac{1}{\sqrt{1+\epsilon^2}}\left(\frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}|11\rangle + \epsilon|22\rangle\right). \tag{4.29}$$

No matter how small ϵ is, this transformation has zero probability of success for any LOCC protocol, because the number of Schmidt coefficients, also known as the Schmidt number or the Schmidt rank, has been increased.

Clearly LOCC operations are in general non-invertible because they can project out Schmidt terms diminishing the Schmidt number of the state whereas the increasing is not possible. Therefore a state $|\psi\rangle$ can be converted into $|\phi\rangle$ by means of LOCC with some probability if and only if the corresponding Schmidt numbers satisfy the relation $n_\psi \geq n_\phi$.

We mentioned when we introduced the Schmidt decomposition that states that can be written using only one Schmidt coefficient, that is, the ones that correspond to unity Schmidt number, are clearly not entangled at all. It is reasonable to assert now that the entanglement of a state characterized by a given Schmidt number is less powerful than that of a state which has a bigger Schmidt number.

So far this LOCC approach has provided us with a tool for classification of entanglement rooted in some physically meaningful criterion. In the following subsection we will introduce a more general scheme with the motivation of presenting a classification of entanglement that works for multipartite systems.

4.3 Stochastic LOCC

It is a natural generalization of LOCC to consider stochastic local quantum operations assisted by classical communication (SLOCC) [16]. By letting the protocol be successful just stochastically, instead of requiring to be successful at each instance as the LOCC protocol does, it is not imposed that the final state has to be achieved with certainty. This loosened version of LOCC is also known as *filtering operation* [15].

This method consists of several rounds of LOCCs, which are dependent on previous measurement results. Success is possible if and only if one of the branches does the job of getting the final state $|\phi\rangle$ from the original one $|\psi\rangle$ and then it is said that $|\psi\rangle$ and $|\phi\rangle$ are equivalent under SLOCC. In other words, interconversion is not asked to be deterministic, in the sense that probability of succeeding is not required to be 1.

SLOCCs are best understood within the formalism of quantum operators described by Karl Kraus in [17]. These operators permit us to describe quantum measurements in a more general way than the von Neumann scheme does. We begin by considering two systems: the target, that is, the system we want to measure, and a second ancillary system or probe [18]. These systems have dimensions M and N respectively and the interaction between them is described by a unitary operator acting on both that can be written as

$$U = \sum_{nn'mm'} u_{nm,n'm'} |nm\rangle\langle n'm'| = \sum_{nn'} |n\rangle\langle n'| \otimes A_{nn'}$$

$$= \begin{pmatrix} A_{11} & A_{12} & ... & A_{1N} \\ A_{21} & A_{22} & ... & A_{2N} \\ ... & ... & ... & ... \\ A_{N1} & A_{N2} & ... & A_{NN} \end{pmatrix}, \tag{4.30}$$

where $|m\rangle$ corresponds to the target and $|n\rangle$ to the probe, and the sub-blocks $A_{nn'} = \sum_{mm'} u_{nm,n'm'} |m\rangle\langle m'|$ are $M \times M$ matrices. Due to unitarity

$$\sum_n A_{ni}^\dagger A_{ni} = B_{ii} = \mathbb{I} \tag{4.31}$$

for every sub-block B_{ii} in $U^\dagger U = \mathbb{I}$ and we can then write $A_n \equiv A_{ni}$ and $A_n^\dagger \equiv A_{ni}^\dagger = A_{in}^*$, so that the previous restriction reads

$$\sum_n A_n^\dagger A_n = \mathbb{I}. \tag{4.32}$$

Given that the unitary operator that governs the interaction between the target and the prove can be any one that acts in the joint system, the probe

may start in the state $|0\rangle$ without loss of generality:

$$\rho_{tot} = |0\rangle\langle 0| \otimes \rho. \tag{4.33}$$

The action of the unitary leads to

$$\begin{aligned}\rho_U =& U\rho_{tot}U^\dagger = \left(\sum_n |n\rangle\langle 0| \otimes A_n\right)(|0\rangle\langle 0| \otimes \rho)\left(\sum_n |0\rangle\langle n| \otimes A_n^\dagger\right) \\ =& \sum_n |n\rangle\langle n| \otimes A_n\rho A_n^\dagger.\end{aligned} \tag{4.34}$$

By allowing the target and the probe to interact, we have let them to get correlated, and now, a von Neumann measurement on the probe will provide information about the target. The action of the projector $P_n = |n\rangle\langle n| \otimes \mathbb{I}$ for example, after tracing out the probe, gives

$$\begin{aligned}\rho'_n =& \frac{\mathrm{tr}_{probe}(P_n\rho_U P_n)}{C} = \frac{A_n\rho A_n^\dagger}{\mathrm{tr}(A_n\rho A_n^\dagger)} = \frac{A_n\rho A_n^\dagger}{\mathrm{tr}(|n\rangle\langle n| \otimes A_n\rho A_n^\dagger)} \\ =& \frac{A_n\rho A_n^\dagger}{\mathrm{tr}(P_n\rho_U P_n)} = \frac{A_n\rho A_n^\dagger}{p_n},\end{aligned} \tag{4.35}$$

where C is a normalization constant that turns out to be the probability p_n of finding the probe in the state $|n\rangle$.

If we do not know the outcome of the measurement, however, all we can say is that the state of the target is going to be

$$\rho' = \sum_n p_n\rho'_n = \sum_n A_n\rho A_n^\dagger, \tag{4.36}$$

that is, an averaging over all the possible results of a von Neumann measurement in the probe. This last form is called the operator-sum representation or the Kraus representation of ρ', and the operators A_n are known as Kraus operators.

Now that we have introduced this very useful formalism of measurement operators, we are ready to tackle the SLOCC scheme and its close relation to invertible local operators (ILOs) [19]. We shall consider first the Schmidt normal form of a general state $|\psi\rangle$ in a bipartite system made of a qutrit and a qubit:

$$(U_A \otimes U_B)|\psi\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle. \tag{4.37}$$

If we apply now the operation given by

$$A \otimes \mathbb{I}_B = \left(\sum_{i=0}^{2} |\lambda_i\rangle\langle i|\right) \otimes \left(\sum_{i=0}^{1} |i\rangle\langle i|\right), \tag{4.38}$$

where A acts on the qutrit and $\mathbb{I}_B$ on the qubit, then the reduced density matrices will be changed in the following way:

$$\rho_A^\psi = \begin{pmatrix} \cos^2\theta & 0 & 0 \\ 0 & \sin^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow A\rho_a^\psi A^\dagger = \cos^2\theta|\lambda_0\rangle\langle\lambda_0| + \sin^2\theta|\lambda_1\rangle\langle\lambda_1|, \tag{4.39}$$

$$\rho_B^\psi = \begin{pmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix} \rightarrow \mathbb{I}_B\rho_b^\psi\mathbb{I}_B^\dagger = \begin{pmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix}. \tag{4.40}$$

So it follows that for the states $|\psi\rangle$ and $|\phi\rangle$ such that

$$|\phi\rangle = (A \otimes \mathbb{I}_B)(U_A \otimes U_B)|\psi\rangle, \tag{4.41}$$

the ranks of the corresponding reduced density matrices satisfy

$$r(\rho_A^\phi) = r(\rho_B^\phi) \leq r(\rho_A^\psi) = r(\rho_B^\psi). \tag{4.42}$$

This expression is in fact a general one for arbitrarily big bipartite systems, and for the multipartite case, with parties $A, B, ..., N$, the generalization reads as follows:

$$r(\rho_k^\phi) \leq r(\rho_k^\psi), \tag{4.43}$$

where $|\phi\rangle = (A \otimes B \otimes ... \otimes N)|\psi\rangle$ and $k = A, B, ..., N$. This is because the operation can be seen as a composition of the local operators $A \otimes \mathbb{I}_{B...N}$ and $\mathbb{I}_A \otimes (B \otimes ... \otimes N)$ and similarly for the other parties. The probability of success of the composition will have the form of a product of probabilities for each step $p_A p_B ... p_N$.

Now, if the operators $A, B, ..., N$ are invertible, then $|\psi\rangle = (A^{-1} \otimes B^{-1} \otimes ... \otimes N^{-1})|\phi\rangle$, that is, the operation can be reversed locally and then both pure states can be reached from each other using SLOCC. We say that $|\psi\rangle$ and $|\phi\rangle$ are equivalent under SLOCC. Also as $r(\rho_k^\phi) \leq r(\rho_k^\psi)$ and $r(\rho_k^\phi) \geq r(\rho_k^\psi)$, it follows that $r(\rho_k^\phi) = r(\rho_k^\psi)$. This is an important result. It means that SLOCC protocols, unlike LOCC, do conserve the local ranks of pure states.

Let us consider the simplest transformation of this kind, that is, $A \otimes \mathbb{I}_{B...N}$ with A invertible. We can write then the following Schmidt decompositions for the initial and final states:

$$|\psi\rangle = \sum_{i=1}^{n} a_i^\psi |i\rangle_A |\mu_i\rangle_{B...N}, \tag{4.44}$$

$$|\phi\rangle = \sum_{i=1}^{n} a_i^{\phi} (U_A |i\rangle_A) |\mu_i\rangle_{B\ldots N}, \tag{4.45}$$

where the local unitary U_A relates the local basis for both states in Alice's m-dimensional part. It is clear then that the operator A should be of the form

$$A = U_A \left(\sum_{i=1}^{n} \frac{a_i^{\phi}}{a_i^{\psi}} |i\rangle\langle i| + \sum_{i=n+1}^{m} |\lambda_i\rangle\langle i| \right). \tag{4.46}$$

In this expression the vectors $|\lambda_i\rangle$ play no role and therefore we can write $|i\rangle$ instead:

$$A = U_A \left(\sum_{i=1}^{n} \frac{a_i^{\phi}}{a_i^{\psi}} |i\rangle\langle i| + \sum_{i=n+1}^{m} |i\rangle\langle i| \right). \tag{4.47}$$

This is a great step, because it makes A diagonal and therefore invertible, because all the diagonal elements are nonzero. The general case, again, corresponds to composing this operation with $\mathbb{I}_A \otimes B \otimes \mathbb{I}_{C\ldots N}$, for which the same argumentation can be applied, and so on and so forth. So this is the prove that if $|\psi\rangle$ and $|\phi\rangle$ are SLOCC equivalent, then the operator relating them can always be chosen to be invertible.

Summarizing, two pure multipartite states are SLOCC equivalent if and only if they are related by an invertible local operator (ILO).

The consideration of the value of the ranks of the reduced density matrices, or local ranks, has proved vital for an understanding of SLOCC protocols. Next we shall see their usefulness for the classification of multipartite entanglement. Let us present the case of non-entangled and bipartitely entangled or biseparable three qubit states first.

4.4 SLOCC classification for non-genuine tripartite entanglement

Using the fact that local ranks of multipartite states are invariant under ILOs we can distinguish four classes or families that are inequivalent under SLOCC, when talking about states that do not have genuine three partite entanglement.

First of all, we present the non-entangled or separable state, that is, the one that can be taken into

$$|\psi_{a-b-c}\rangle = |000\rangle, \tag{4.48}$$

using some convenient local unitaries. We get that

$$\rho_a = \rho_b = \rho_c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.49}$$

so that $r(\rho_a) = r(\rho_b) = r(\rho_c) = 1$.

The representatives of the other three classes are the bipartitely maximally entangled

$$|\psi_{ab-c}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)|0\rangle, \tag{4.50}$$

$$|\psi_{ac-b}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)|0\rangle, \tag{4.51}$$

$$|\psi_{bc-a}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)|0\rangle, \tag{4.52}$$

states, for which respectively $r(\rho_c) = 1$, $r(\rho_b) = 1$ and $r(\rho_a) = 1$, being the other local ranks equal to 2, as they correspond to the maximally entangled qubits in each of the cases.

Two, out of the three, local ranks are different from one state to another and therefore we conclude that the four states belong to four inequivalent classes under SLOCC.

Note that other states within the biseparable classes can be obtained from one of the three representatives by means of LOCC with total certainty, in the same manner we did for bipartite states, and the separable state can be obtained from any of them.

4.5 SLOCC classification for symmetric 3-qubit states

The $|GHZ\rangle$ and $|W\rangle$ states we introduced at the beginning of this section were both symmetric with respect to the permutation of the qubits. On the other hand, the symmetric subspace has the advantage of a lower increasing of its dimension with the number of parties: if we consider a 1/2 spin compound system formed of n spins, the $|S, M\rangle$ or Dicke states are simultaneous eigenstates of the collective spin operators S^2 and S_z. The states having the highest value of the total angular momentum quantum number $S = n/2$ are symmetric with respect to permutation, and form a subset of all 2^n Dicke states [20]. As there are $2S + 1$ possible values of M for every value of S, it follows that the dimension of the symmetric subspace is $n + 1$.

Finally, and most importantly, limiting ourselves to the symmetric subspace permits us to act with the same invertible local operator A on each of the qubits, because then it is sufficient to look for a symmetric ILO [21]:

$$|\phi_S\rangle = (A \otimes ... \otimes A)|\psi_S\rangle, \tag{4.53}$$

where $|\phi_S\rangle$ and $|\psi_S\rangle$ are symmetric states.

For 3 qubits, the 4 linearly independent symmetric Dicke states are usually written as follows:

$$\begin{aligned}
\left|\frac{3}{2},\frac{3}{2}\right\rangle &= |000\rangle, \\
\left|\frac{3}{2},\frac{1}{2}\right\rangle &= \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle), \\
\left|\frac{3}{2},\frac{-1}{2}\right\rangle &= \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle), \\
\left|\frac{3}{2},\frac{-3}{2}\right\rangle &= |111\rangle,
\end{aligned} \tag{4.54}$$

where in the right hand side we already substituted the individual angular momentum projection values by their qubit notation counterparts. These states form a basis on the 3-qubit symmetric subspace and therefore any state belonging to it can be expressed as a linear combination of them:

$$|\phi_S\rangle = w\left|\frac{3}{2},\frac{3}{2}\right\rangle + x\left|\frac{3}{2},\frac{1}{2}\right\rangle + y\left|\frac{3}{2},\frac{-1}{2}\right\rangle + z\left|\frac{3}{2},\frac{-3}{2}\right\rangle, \tag{4.55}$$

where w, x, y and z are complex coefficients.

This scheme suggested us a straightforward technique to find different SLOCC families. Indeed, using the expression 4.53, where we parametrize the invertible operator as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{4.56}$$

so that the inverse is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \tag{4.57}$$

we get a system of equations relating w, x, y and z to a, b, c and d for a given $|\psi_S\rangle$. Taking $|\psi_S\rangle_1 = |GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ and $|\psi_S\rangle_2 = |W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$, we have found that the two systems of equations we get are incompatible, that is, there do not exist w, x, y and z coefficients such that they can be written in terms of the a_1, b_1, c_1 and d_1 as well as the a_2, b_2, c_2 and d_2, at the same time:

$$
\begin{aligned}
w &= \frac{1}{\sqrt{2}}(a_1^3 + c_1^3), \\
x &= \frac{1}{\sqrt{2}}(a_1^2 b_1 + c_1^2 d_1), \\
y &= \frac{1}{\sqrt{2}}(a_1 b_1^2 + c_1 d_1^2), \\
z &= \frac{1}{\sqrt{2}}(b_1^3 + d_1^3),
\end{aligned} \tag{4.58}
$$

$$
\begin{aligned}
w' &= \sqrt{3} a_2^2 c_2, \\
x' &= \frac{1}{\sqrt{3}}(a_2^2 d_2 + 2 a_2 b_2 c_2), \\
y' &= \frac{1}{\sqrt{3}}(b_2^2 c_2 + 2 a_2 b_2 d_2), \\
z' &= \sqrt{3} b_2^2 d_2.
\end{aligned} \tag{4.59}
$$

$$
\implies w \neq w',\ x \neq x',\ y \neq y',\ z \neq z'. \tag{4.60}
$$

We performed the same test with $|\psi_S\rangle_1 = |000\rangle$ and $|\psi_S\rangle_2 = |111\rangle$ and this time we got a relation between a_1, b_1, c_1 and d_1, and a_2, b_2, c_2 and d_2. This is how it should be, of course, because $|000\rangle$ and $|111\rangle$ are local unitarily equivalent and local unitaries are a special case of SLOCC operations, therefore belonging to the same SLOCC family, namely, the family of separable states.

Chapter 5

Conclusions

In this final chapter, we would like to summarize some important aspects of entanglement we learned during this theoretical study.

- The Schrödinger equation not only admits product states, but also superpositions of them, leading to correlations between observables that defy a classical reasoning. This does not mean, however, that all superposition states are entangled. Some turn out to be separable. This concept of separability is the one on which the definition of entanglement lies.

- To tell whether or not a state is separable, there does not exist a general procedure. However different criteria exist for some particular cases. For pure states it is enough to look for the reduced density matrices and see if they represent a mixed state. Then the state is entangled. For mixed states of two qubits or a qubit and a qutrit, a necessary and sufficient condition exists, which is given by the PPT criterion.

- For three qubits the variety of entangled states grows. If the LOCC scheme confirms us that there are different degrees of entanglement in the bipartite case, being the Bell states the maximally entangled ones, SLOCC shows that in the tripartite case, although no such thing as the maximally entangled state exists, a classification based on the local ranks can be envisaged: separable states have local ranks which equal all to 1; non-totally separable but non-genuinely entangled tripartite states have one local rank equalling to 1; genuinely entangled tripartite states such as the W or the GHZ have local ranks all equalling to 2.

- However, W and GHZ states are not reachable from each other by means of ILOs and therefore are representatives of distinct SLOCC classes.

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