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D-Wave quantum computer

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Contents

1 Introduction

Optimization problems and particularly combinatorial optimization problems are very important nowadays. In fact, as we will see in this work, the current applications of these problems are huge and in some cases extremely important in the creation of modern technology such as microchip architecture design. Nevertheless, these problems usually demand too much computational effort and time as their size grows. Therefore, there are problems of some size that we cannot compute in reasonable time.

Secondly, the microchips that are being built right now are reaching the point where the law of Moore[1] is no longer valid. That is because transistors are beginning to have nanometric size where quantum mechanics are the laws that govern. Thereupon, the transistors begin to fail because of undesired quantum effects, e.g., quantum tunneling of electrons over the insulator. Thus, we are reaching the limit scale of the transistors. In order to solve this problem that is currently happening there are some approaches trying to refine the computation techniques that are completely developed nowadays. These include parallel processing, or the use of more than one processor in parallel to make the calculations faster, seen in the processors of modern personal computers or in the architecture of the supercomputers.

Nonetheless, there is other approach to this problem that consists in changing radically the paradigm and tries to use the quantum mechanical effects that are so harmful to transistors, to make quantum computers[2]. This approach was first suggested by R. Feynman in 1982[1] stating that quantum computers would simulate much better quantum systems than classical computers. However, there was not a big movement in the field until 1994 when Peter Shor [2] showed that an algorithm exists to factorize numbers whose computation time has an exponential gain over the best known classical algorithm. This was shocking because this algorithm can break any current public key code in cryptography in just a few seconds creating a great interest in quantum computation and its possible applications.

There have been many models proposed to do quantum computation exemplified as topological quantum computation, gate based computation, and adiabatic quantum computation. On one hand, the second one is the best attempt to make a universal quantum computer [3]. On the other hand, the easiest architecture to scale nowadays are superconducting circuits used for quantum adiabatic computation as was shown by M.W. Johnson et al. [3]. However, it was not clear that adiabatic quantum computation was equivalent to gate based quantum computation. This fact was demonstrated by D. Aharonov et al. [4] in 2007.

Recently, an experiment using superconducting qubits to simulate adiabatic quantum computation digitally [5] by a collaboration between Google and a research group from this university, QUTIS, has been performed. This is a milestone because this method combines the advantages of both methods. Therefore, it is very interesting and necessary to have strong knowledge about quantum annealing and the machine that uses this technique to solve optimization problem calculations.

[1] It states that the number of transistors inside a microchip are doubling every two years. This shows the tendency of the hardware market.

[2] Programmable physical systems that have quantum-mechanical properties such as quantum superposition, quantum fluctuations and quantum tunneling.

[3] Computer that can be programmed to do any kind of calculation.

In this work, I will start doing a review of spin models used in condensed matter physics and some of the best known optimization problems. Then, I will explain the simulated annealing method and its quantum counterpart along with the simulation of quantum annealing. After that, I will analyze the D-Wave 2X machine characteristics, its mechanism and review the most recent comparisons with the simulated annealing and the algorithms working in the fastest classical supercomputers. Last but not least, I will make a basic simulation to show how simulated annealing and quantum annealing work.

2 Optimization problems

Optimization problems consist in searching for the optimal solution, minimum or maximum, of a function usually called cost function. In addition, there are two classes of optimization problems depending on the variables of the cost function: discrete, also known as combinatorial optimization problems (COP), and continuous. There are many COPs such as Traveling Salesman Problem(TSP), Constraint Satisfaction Problem(CSP), Nurse Scheduling problem(NSP),etc Every possible solution of a COP can be mapped using a graph theory such that it can be reduced to a matrix.

2.1 Traveling Salesman Problem

Even though it is unknown who was the first person to enunciate the problem, W.R. Hamilton and T. Kirkman were the first in the XIX^{th} century to formulate it mathematically. The problem is to find the path among N cities that a person has to travel in order to minimize the length. Each city can be visited just once. In principle, the total possible paths are $\frac{(N-1)}{2}!$ so as N grows it is not viable to check all the possible paths or use the algorithm of going always to the nearest town. On the top of that, the problem was proved to be NP-hard[4] by R.M. Karp[6] in 1973. In particular, the best known algorithm scales exponentially with the number of cities $O(n^2 2^n)$. As a matter of fact, the TSP is a very interesting problem given the many practical applications it has, for example route planning, manufacture of microchips and DNA sequencing.

2.2 Boolean satisfiability(SAT)

Another interesting COP is to determine if a boolean formula[5] is true or false, known as boolean satisfiability problem(SAT), a subclass of CSPs. If you have K clauses separated by boolean operators, to determine if they are true or not is called K-SAT problem. This problem is hard to solve, and the best known algorithm for 3-SAT for example scales as $O(2^{0.386n})$. This was one of the first problems that was proved to be NP-complete. Finally, this problem is used in areas such as Artificial Intelligence(AI), automatic test pattern, and electronic design among others.

[4] An NP problem is NP-hard if you can transform any NP problem to your problem in polynomial time.
[5] A formula built with logical clauses and operators AND, OR and NOT.

Imagine we have N boolean variables[6] named as $x_1, x_2, ..., x_N$ and we have clauses made of K boolean variables, in this case $K = 3$, named as C_l defined as:

$$C_l = (x_i \vee x_j \vee x_k) \ , \tag{1}$$

where x_i, x_j and x_k are taken randomly. 3-SAT problem tries to answer the question that given M clauses the conjuction($\wedge$) of them, i.e., the following equation is true of false ,i.e., 1 or 0.

$$P = C_1 \wedge C_2 \wedge ... \wedge C_{M-1} \wedge C_M \ . \tag{2}$$

3 Quantum annealing

Quantum annealing is a computation technique that tries to find the global minimum of a given function. The subsection 3.1 gives a brief review of spin models whereas 3.2 will explain the method of simulated annealing, a purely classical approach in which quantum annealing is inspired. In 3.3 I will describe the method of quantum annealing and how we can use it to solve the two examples of optimization problems I explained in 2. Finally, I will review a couple of methods to simulate quantum annealing protocols in 3.3.5.

3.1 Spin models

Spin models were created in order to explain the magnetic behavior of materials, such as ferromagnetism and antiferromagnetism. The most accurate and general model is the Heisenberg model[7]. Suppose that there is a one-dimensional chain of N atoms where each atom can have its spin pointing in any direction. Furthermore, we will consider a dipole-dipole interaction between each pair of spins. Therefore, the Hamiltonian will be :

$$H = -\sum_{i<j}^{N} J_{ij}(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z) - h \sum_{i}^{N} (\sigma_i^x + \sigma_i^y + \sigma_i^z) \ , \tag{3}$$

where J_{ij} are the interaction constants between the spins, h is the external magnetic field and σ_i^x, σ_i^y and σ_i^z are the Pauli matrices of the i^{th} lattice point.

It turns out that this model is too complicated to solve analytically and it is interesting to consider a simpler model that has analytical solutions while maintaining the qualitative aspects of the Heisenberg spin model. This model, called Ising spin model, was proposed in 1925[8]. The main difference of this model with the Heisenberg model is that the spin must point along the z axis such that the Hamiltonian reads now,

$$H = -\sum_{i<j}^{N} J_{ij}\sigma_i^z \sigma_j^z - h \sum_{i}^{N} \sigma_i^z \ , \tag{4}$$

[6]Variable that can only have two outcomes 1 and 0 representing if the variable is true or false respectively.

where σ_i^z is the Pauli matrix along the z axis.

In spite of the fact that in the one dimensional model there is no phase transition, when one considers bidimensional or tridimensional lattices there are transitions between the ordered, ferromagnetic or antiferromagnetic phases, and disordered states, known as spin glass phases due to the similarities with glasses which are distinguished for being amorphous. In these lattices, there is usually a high degeneracy in the ground state. Let us consider an easy example analized by G.H. Wannier[9] shown in Fig 1. There are three antiferromagnetic atoms per triangular grid. If two of the atoms have opposite spins to minimize the energy between them, the third atom cannot align with the other two atoms simultaneously and there is a sixfold degeneracy in the ground state.

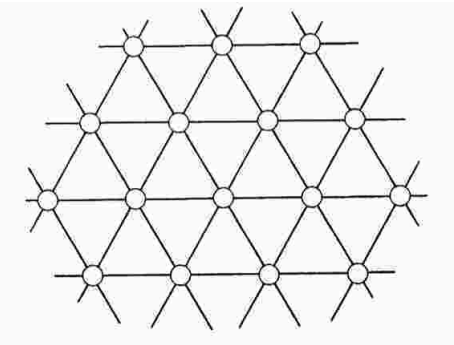

Figure 1: Frustation in a triangle grid. Ref.: [9]

This phenomenon is called geometrical frustration and it is common in bidimensional and tridimensional lattices. In addition, geometric frustration usually makes it very hard to find the ground state of the sample. This was demonstrated by F. Barahona[10] showing that finding the ground state of a frustrated Ising spin glass was an NP-complete problem [7].

Moreover, the potential energy landscape (PEL) is usually very wrinkled. This can cause that the system is trapped in a local minimum instead of the global minimum as it should be. Additionally, this effect called quenching breaks the ergodicity[8] because the system cannot travel anymore through the whole PEL.

Finally, there exist two possible ways to make a model for the interactions between the spins in spin glasses, J_{ij}, considering all the possible interactions, Sherrington-Kirkpatrick[11] model, and supposing that there are just interactions between nearest neighbors, Edward-Anderson[12] model. Nevertheless, both models use a Gaussian distribution as the density of the interaction constants, see Eq. (5). In contrast, if we want to use these models to program difficult problems in our spin glass, we must have the freedom to choose any possible J_{ij} configuration.

$$\rho(J_{ij}) = A \exp \frac{-J_{ij}^2}{2J^2} \tag{5}$$

[7]This is a classification method of problems used in computation theory. In NP problems the time it needs for the best known algorithm to solve them does not scale as a polynomial of the system size. Besides, if you can transform any NP problem to that problem and check the solution in polynomial time then the problem is NP-complete.

[8]The system passes over all possible microstates of the phase space.

3.2 Simulated annealing

In order to solve the problems I introduced in the previous section, S. Kirkpatrick et al. [13] proposed a method using statistical mechanics. The main idea of the method was inspired by a metalurgy technique called annealing used as a treatment to reduce the number of dislocations, defects that are created because of the planes of the monocristals not matching. In this technique the metal sample is heated up until it reaches a temperature around the fusion point. Afterwards, the temperature is mantained constant to let the monocrystal reconfigure. Finally, the temperature is descended very slowly to ambient temperature leaving the metal ordered.

In simulated annealing, we write our optimization problem, the cost function, as a potential energy of a many-body system. Then, the temperature of the system is increased such that the system can jump through the local maxima in order to reach the global minimum. Subsequently, the temperature is decreased slowly, allowing the system to arrive at a steady state every time the temperature is descended. In fact, it was demonstrated by S. Geman and D. Geman[14] in 1984 that the temperature of each step of the simulation has a bound:

$$T = \frac{c}{\ln(k+1)} \ , \tag{6}$$

where T is the temperature parameter of the simulation, c a constant to be determined and k is the time step number.
Therefore, we can say that there exists a bound for the annealing temperature using continuous time:

$$T \propto \frac{1}{\ln(t)} \ , \tag{7}$$

taking into account the difficulty of making a real physical system to have the PEL equal to the cost function we want to solve. As a consequence, we need to simulate the system using the Metropolis-Hasting method that has the following steps for each iteration,

1. The algorithm starts in a state s with energy E_s
2. Call a random number (r) using number obtained by a pseudorandom number generator
3. Look if the neighbors (s') are accesible:
   - if $\Delta E \equiv E_{s'} - E_s \leq 0$ then the state is accesible.
   - if $\Delta E > 0$ then if the Boltzmann factor, $p(\Delta E) \equiv e^{\frac{\Delta E}{K_b T}}$, is larger than r, i.e., $p(E_s) > r$ the state is accesible
   - In all the other cases the states are unaccesible
4. Continue to the next time step $t' = t + \tau$ where τ is the time discretization
5. Change the temperature $T' = T - \alpha(t)$[9]

[9]Note that the change in temperature must be slow to be able to reach the global minimum.

This method is easy to code while it can be applied to any kind of cost function and it statistically finds the solution: it will reach the minimum when $t \to \infty$. Notwithstanding, it has some inconvenience, e.g, it behaves poorly if the cost function is expensive even if it finds the optimal answer, and nothing guarantees that the solution is the correct, and most optimal. What is more, if the system is stuck in a local minimum because of the high barriers it has around, the method is extremely uneffective. After all, there could be a method that still uses the idea of a slowly changing field, like the temperature field used in simulated annealing, but without the problem of getting stuck in the local minima.

3.3 Quantum annealing

3.3.1 Transverse field Ising model

It has been shown in 3.2 that, despite being a good general optimization method, simulated annealing is worthless if the PEL has very peaked and narrow maxima near a local minima. In order to overcome this problem, H.Nishimori and T. Kadowaki [15] in 1998 proposed to use a transverse magnetic field to create quantum fluctuations that can make the system pass through the high and narrow peaks where SA usually gets stuck using quantum tunneling, as we can see in Fig 2.

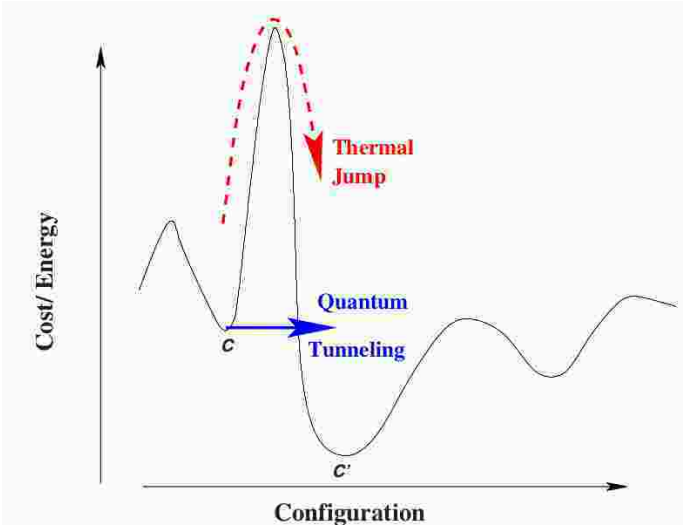

Figure 2: Advantage of quantum tunneling over thermal jump. Ref.:[16]

The model that H. Nishimori and T. Kadowaki proposed is an Ising spin model but with a time-dependent magnetic field in the x direction such that the Hamiltonian is the following,

$$H = H_p + H_i = -\sum_{i<j}^{N} J_{ij}\sigma_i^z\sigma_j^z - h\sum_{i}^{N}\sigma_i^z - \Gamma(t)\sum_i \sigma_i^x \ , \tag{8}$$

where H_p is the problem Hamiltonian, H_i the changing Hamiltonian and $\Gamma(t)$ a time dependent magnetic field.

The key feature of this model is that the commutator $[H, \sigma_i^z] \neq 0$ creates interferences between the eigenstates of the system that enable it to tunnel through the barriers. However, the probability of tunneling decreases as the potential that the system needs to cross becomes wider.

3.3.2 Quantum tunneling

To understand this feature we can analyze a toy model to study the qualitative behaviour of a quantum system. In this model, we have the potential barrier seen in Fig. 3 with height V_0 and width $d = x_2 - x_1$.

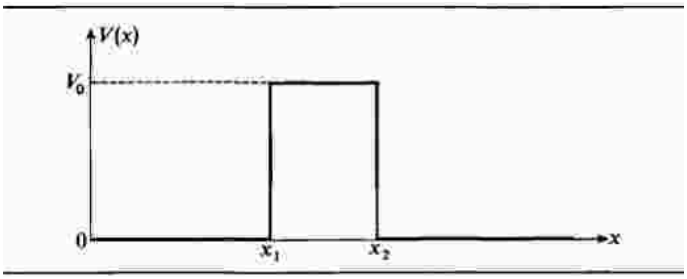

Figure 3: one dimensional potential barrier. Ref.:[17]

We consider a particle of mass m with energy $E < V_0$ passing through this potential energy, and we want to know the transition probability[10] of the particle $T \equiv \frac{|\psi_3|^2}{|\psi_1|^2}$ where ψ_1 and ψ_3 are the wavefunctions in the left and right side of the barrier. Therefore, it is neccesary to employ the equation of motion of quantum systems, Schrödinger equation, whose general form is,

$$H\psi = i\hbar\frac{\partial\psi}{\partial t} \ . \tag{9}$$

In this case, the Hamiltonian is $H = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0(\theta(x_1 - x) - \theta(x_2 - x))\psi$ and Eq.(9) takes the form,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0(\theta(x_1 - x) - \theta(x_2 - x))\psi = i\hbar\frac{\partial\psi}{\partial t} \ . \tag{10}$$

The particle will behave like a plane wave in the left hand side of the barrier such that $\psi_1 = Ae^{-ikx} + Be^{ikx}$ where $k = \sqrt{\frac{2mE}{\hbar^2}}$. In the right hand side of the barrier the particle is a free particle as well $\psi_3 = Ce^{-ikx} + De^{ikx}$. Using normalization and continuity conditions over the whole space, the transition probability is,

$$T = \frac{1}{1 + \frac{V_0^2 \sinh^2\Delta}{4E(V_0 - E)}} \ , \tag{11}$$

where $\Delta \equiv d\sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$.

From Eq.(11) we can see that if we mantain V_0 constant and increase d, the transition probability becomes smaller and in the limit,

[10]The probability that the particle goes through the barrier.

$$\lim_{d\to\infty} T = 0 \ . \tag{12}$$

3.3.3 Adiabatic theorem and adiabatic quantum computation

In 1998, there was the first experimental confirmation that quantum fluctuations due to a transverse magnetic field can make the system relax faster than using thermal ones, i.e., that quantum annealing could beat simulated annealing solving optimization problems. The experiment was carried out by J.Brooke et al. [18] using a $LiHo_{0.44}Y_{0.56}F_4$ crystal. Later, E. Fahri et al. [19] proposed to use a theorem employed in quantum mechanics called adiabatic theorem, discovered by V. Fock and M. Born in 1929, to ensure that the system reaches its ground state when the annealing schedule finishes. Suppose that we have a Hamiltonian,

$$H(t) = H_0 + W(t) \ , \tag{13}$$

where the time dependent potential is zero at $t = 0$,i.e., $W(0) = 0$.

The system begins in the eigenstate of H_0 called $|0\rangle$. According to the theorem, if $W(t)$ evolves slowly enough to avoid Landau-Zener tunneling, from ground to first excited state, at time t the system will be in ground state of H , i.e., $H|0'\rangle = E_{0'}|0'\rangle$. In principle to ensure that there is no jump to an excited state, the time must be infinite which is unfeasible. Therefore, we can calculate the probability of success of the annealing using time-dependant perturbation theory around the point with smallest gap as M.H.S. Amin et al.[20],

$$P_{success} = 1 - e^{\frac{-t_{annealing}}{t_a}} \ , \tag{14}$$

where $t_{annealing}$ is the annealing time and $t_a \equiv \frac{4\hbar}{\pi}\frac{\langle 0|\frac{dH}{ds}|1\rangle}{g_m^2}\Big|_{s=s'}$, where s' is the state where the gap is the smallest, $|0\rangle$ and $|1\rangle$ are the ground, and first excited state respectively and g_m is a constant strongly dependent of system size.

As a result, we can determine the optimal annealing time to be sure to 100 % that the final state is the ground state and the answer is correct.

In order to use this theorem for computation, it is convenient to use the Ising transverse model to codify our problem as H. Nishimori and T. Kadowaki did and we have to turn it on slowly. The Hamiltonian that illustrates this is the following,

$$H(t) = -A(t)(\sum_{i<j}^{N} J_{ij}\sigma_i^z\sigma_j^z + h\sum_{i}^{N}\sigma_i^z) - B(t)\sum_i \sigma_i^x \ , \tag{15}$$

where $A(0) = 0$ and $B(0) = 1$.

Accordingly, if we want the system to be in the ground state, the initial state has to be (using braket notation),

$$|\psi(t=0)\rangle = |+_1\rangle_x \otimes |+_2\rangle \otimes ... \otimes |+_{n-1}\rangle \otimes |+_n\rangle_x \ , \tag{16}$$

where $|+_i\rangle_x$ is the eigenstate of σ_i^x.

Suppose now that the quantum annealing ends in $T >> 1$, where $A(T) = 1$ and $B(T) = 0$. The adiabatic theorem ensures that $|\psi(t = T)\rangle$ will be the ground state of $H(T)$ such that the energy of the system will be the aimed-for ground state energy.

Finally, D-Wave company, see Neil G. Dickson and Mohammad H. Amin [21], have tried another approach to quantum annealing that consists in running non adiabatically the device and storing the final state. Then, after some iterations they find an optimal solution faster than using the otherwise limited adiabatic theorem.

3.3.4 Solution of optimization problems using quantum annealing

T. Kadowaki [22] was the first one who used quantum annealing to solve optimization problems, in this case TSP in his doctoral thesis. He took every posible configuration of the Ncities and he codified the path with a matrix, T_{ij}, as follows. If the traveller goes from city i to city j $T_{ij} = 1$. In the other cases, $T_{ij} = 0$. We need to make the the matrix symmetric such that we define the symmetric matrix, U_{ij},

$$U_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) \tag{17}$$

Anyhow, to calculate the length of the whole path we have to know the distances between all the cities codified in the matrix, d_{ij}. Accordingly, the total length of the path is:

$$H = \frac{1}{2}\sum_{ij}^{N} d_{ij}U_{ij} \tag{18}$$

If we employ $\sigma_i^z\sigma_j^z = 2U_{ij} - 1$, the total length takes the form of a Hamiltonian:

$$H_{TSP} = \frac{1}{4}\sum_{ij}^{N}(d_{ij} + d_{ij}\sigma_i^z\sigma_j^z) \tag{19}$$

On one hand, the first term is the sum of all paths between the cities. On the other hand, the second term is an Ising spin model where the distances d_{ij} take the role of the interactions between the spins, J_{ij}. Thus, if we solve the Ising spin problem, we will know the length of that particular path. Consequently, if we do this calculation to all possible paths, $\frac{(n-1)!}{2}$, we can find which is the shortest possible path. It may seem a pretty rude approach to the problem. Accordingly, it is convenient to use some better strategy. If our problem, codified by the matrix d_{ij}, has a strong correlation called euclidean[11], it is an optimal strategy to subdivide our problem in smaller ones[12]. In contrast, if our problem does not have that correlation, there is no special method.

Another optimization problem solved using quantum annealing is the 3-SAT problem by G. Santoro et al. [23] in 2005. To solve the problem as we have defined it in section 2.2, we define the spin variable as follows,

$$\sigma_i^z = (-1)^{x_i} \ . \tag{20}$$

Therefore, the energy of a clause C_l is:

$$E_l = \frac{(1 + J_{l,i}\sigma_i^z)(1 + J_{l,j}\sigma_j^z)(1 + J_{l,k}\sigma_k^z)}{8} \ , \tag{21}$$

where $J_{l,i}$, the coupling constant, is -1 if the variables are negated, $\bar{x_i}$, and 1 if not. If $E_l = 0$ the clause is true and $E_l = 1$. As a consequence, we can figure out if P of Eq.(2) is true or false using the following Hamiltonian,

$$H = \sum_l^M E_l = \sum_l^M \frac{(1 + J_{l,i}\sigma_i^z)(1 + J_{l,j}\sigma_j^z)(1 + J_{l,k}\sigma_k^z)}{8} \ . \tag{22}$$

Accordingly, the energy of the system would be 0 if and only if the statement P is true. Moreover, the energy of the system gives you how many false clauses are in P.

3.3.5 Simulated quantum annealing

Until the creation of fully programable chips that use quantum annealing, named as quantum annealers, such as the one I am going to analize, D-Wave $2X^{TM}$, the only possible implementation of the method was via simulations. In order to do them there are some approaches depending on the system size. If the size of the system we want to analyze is small, $N \leq 9$, we can use Schrödinger equation, using real time

[11] For any three cities called i,j and k, the sum of any two of the sides, e.g., d_{ij}, d_{ik} must be larger than the remaining one d_{jk}.

[12] Strategy known as divide and conquer in refence of the quote of Julius Caesar "divide et impera".

or imaginary time evolution, to simulate the quantum annealing as was analized by H. Nishimori and T. Kadowaki [15].They just analize random spins, not optimization problems.

Apart from the fact that there are not many problems that can be codified in the interaction among 9 spins or less, this method has no gain in time over simulated annealing due to the computational effort that Schödinger equation demands to be solved.

Thus, a better and more sophisticated method is needed to simulate the system. There are some possible algorithms as these: Path integral Monte Carlo method (PIMC) and Green's function Monte Carlo method (GFMC).

PIMC aims to simulate the thermodinamics of the system calculating its partition function.,

$$Z(T,\Gamma) = Tr(e^{\beta H}) = \sum_s \langle s|\, e^{\beta H}\, |s\rangle \ , \tag{23}$$

where H is the Hamiltonian of Eq. (8), $\beta \equiv \frac{1}{K_B T}$, K_B is the Boltzmann constant, T is the temperature and s is a configuration of spins, such that we have to sum over all posible spin configurations.Then, it employs the following relation to calculate the internal energy,

$$U \equiv \frac{\partial(\ln Z)}{\partial \beta} \ . \tag{24}$$

This partition function cannot be sampled using Metropolis-Hastings algorithm, because the commutator between the two Hamiltonians is not zero, $[H_p, H_i] \neq 0$. To approximate this partition function, we can use the Suzuki-Trotter formalism [24]. The main idea behind this formalism is to change the dimensions of the system, going from n dimensions to n+1, creating replicas of the system along the extra dimension, called Trotter dimension, that simulates the time dimension. With this trick we can do this change in the partition function keeping the physics unchanged,

$$e^{\beta(H_p+H_i)} = \lim_{P\to\infty} (e^{\frac{\beta}{P}H_p} e^{\frac{\beta}{P}H_i})^P \ , \tag{25}$$

where $\beta \equiv \frac{1}{k_B T}$ and P is the number of replicas along Trotter dimension.

Applying Eq.(25) to Eq.(23) we have ,

$$Z(T,\Gamma) = \lim_{P\to\infty} \sum_{s^1,s^2,\ldots,s^P} e^{\frac{-\beta}{P}\sum_k^N H_i(s^k)} \langle s^1|\, e^{\frac{-\beta}{P}H_p}\, |s^2\rangle \cdots \langle s^P|\, e^{\frac{-\beta}{P}H_p}\, |s^1\rangle \ , \tag{26}$$

where s^k corresponds to the configuration of the k^{th} Trotter replica.

In order to calculate the $\langle s^k |\, e^{\frac{-\beta}{P} H_p} \,| s^{k+1} \rangle$ terms,

$$\langle s^k |\, e^{\frac{-\beta}{P} H_p} \,| s^{k+1} \rangle = C^N e^{\frac{\beta}{P} J^{\perp} \sum s_i^k s_i^{k+1}} \ , \tag{27}$$

where $J^{\perp} \equiv \frac{-\beta}{P} \ln(\tanh(\frac{\beta\Gamma}{P}))$ and C is a constant.

We cannot take infinite Trotter replicas such that we have to take a number of Trotter replicas that can be computed optimally while we try to maintain the maximum similarity with the original partition function. Finally, the partition function is:

$$Z(T, \Gamma) \approx C^{NP} \sum_{s^1} \sum_{s^2} \cdots \sum_{s^P} e^{\frac{-\beta}{P} \Delta} \ , \tag{28}$$

where $\Delta \equiv -\sum_{k=1}^{P} (\sum_{i<j} J_{ij} s^i s^j + J^{\perp} \sum_i s^i s^{i+1})$.

On the whole, there are two kinds of interactions in our system after the change: the ones that we already have between all the spins of our n dimensions J_{ij} and the interaction between the n dimensions with the Trotter dimension $J^{\perp}$.

GFMC[13] aims to simulate the time evolution of the system using time evolution operators with imaginary time[14] ,

$$U = e^{H(t)\ t} \ . \tag{29}$$

In order to simulate the time evolution, we must make time discretization with interval $\Delta\tau$. In every step the Hamiltonian changes a bit, and the change occurs stepwise, letting the state of the system change obeying the following expression from step $n-1$ to step n:

$$|\psi(\tau_n)\rangle = e^{H_n \tau_n} \,|\psi(\tau_{n-1})\rangle \ , \tag{30}$$

where $\tau_n = \tau_{n-1} + \Delta\tau$ and H_n is the Hamiltonian at the n^{th} step of the simulation. Then, the complete evolution of the system must be:

$$|\psi(t)\rangle = e^{H_n t_n} e^{H_{n-1} t_{n-1}} ... e^{H_2 t_2} e^{H_1 t_1} \,|\psi(0)\rangle \ , \tag{31}$$

where $|\psi(0)\rangle$ is the initial state of the system. We now make a first order approximation of the exponential in Eq. (30),

[13] In this method the temperature is zero always.

[14] In the explanation of this method $\hbar = 1$.

$$|\psi(\tau_n)\rangle = [1 - \Delta\tau(H_n - E_T)]\,|\psi(\tau_{n-1})\rangle \ , \tag{32}$$

where E_T is an approximate estimation of the energy of the ground state that is updated every time we run the algorithm to obtain a more accurate solution. This estimation of the energy helps the method to converge faster. If we apply $\langle x'|$ and $\langle x|$ to obtain the wave functions of the n^{th} and the $(n-1)^{th}$ step we obtain,

$$\psi_n(x') = \sum_x G^{(\Gamma)}_{x',x}\psi_{n-1}(x) \ , \tag{33}$$

where $G^{(\Gamma)}_{x',x} = \langle x'|\, e^{H_n\tau_n}\,|x\rangle \approx (1+\Delta\tau)\delta_{x'x} - \Delta\tau\,\langle x'|\, H_n\,|x\rangle$.

$G^{(\Gamma)}_{x',x}$ has similarities with the matrix that represents a process called Markov chain.[15] However, there are some differences that can cause problems:

1. In general systems, the elements of $G^{(\Gamma)}_{x',x}$ can be negative, which causes a great problem known as sign problem.
2. The sum of the elements of each row of $G^{(\Gamma)}_{x',x}$ does not have to be 1, $\sum_{x'} G^{(\Gamma)}_{x',x} \neq 1$, which makes that the matrix elements do not represent probabilities and the sum of all of them is not 1.

On one hand, the first problem disappears in the Ising spin glass case because in this case all elements of $G^{(\Gamma)}_{x',x}$ must be positive or at least 0. On the other hand, the second problem remains and forces us to reconstruct the matrix $G^{(\Gamma)}_{x',x}$ in every step before doing the Markov chain step in this way:

$$G^{(\Gamma')}_{x',x} = \frac{G^{(\Gamma)}_{x',x}}{b_x} \ , \tag{34}$$

where $b_x \equiv \sum_{x'} G^{(\Gamma)}_{x',x}$ is a different constant for every row.

With this element and a weight $w$to have record of the b_x, the Markov process of the matrix $G^{(\Gamma)}_{x',x}$ has the following steps,

1. We update the matrix $p_{x',x}$.
2. We obtain the new configuration x' using the matrix $p_{x',x}$.

[15]Stochastic process where the probability of some event is just conditional of what happened in the previous event. The information of the conditional probabilities is stored in a matrix $p_{x,x'} \equiv p(x|x')$.

3. We update the weight $w' = w\ b_x$.

This random walker still has a significant problem which is the divergence of the weight or its convergence to zero very fast. This can be fixed using more than one random walker and after some steps you can calculate the mean value of the weights to balance them, the ones that go to zero with the weights that go to infinity.

Finally, it is extremely useful to use a trial wavefunction ψ_T to modify the Green function in order to find the solution faster and more accurately,

$$G^{\bar{\Gamma}}_{x,x'} = \psi^t_T G^{\Gamma}_{x,x'} \psi_T \ . \tag{35}$$

4 D-Wave 2X

$D-Wave\ 2X^{TM}$ is the newest quantum computer manufactured by D-Wave Systems company. It is the only quantum computer in the market nowadays. Firstly, in 4.1 I will make a review of the history of the company focusing on the different prototypes of quantum annealers. Then, in 4.2 I will explain how the chips are built and their architecture and I will also analyze the physical characteristics of the chip. After that, in 4.3 I will briefly analyze the best proof that $D-Wave2X^{TM}$ works using quantum mechanics, namely the entanglement among the individual elements of the machine ,i.e., the qubits[16]. Later, I will describe how the noise affects quantum systems in general and quantum annealers in particular in 4.4. In 4.4 I will also show some ways that try to correct the errors created by the noise. Then, in 4.5 I will discuss new possible architectures of quantum annealers that can solve some limitations of the current annealers. Finally, I will explain the most important aspects of quantum computation and why these machines were created in 4.6, namely to beat supercomputers in solving combinatorial problems.

4.1 History of D-Wave Company

D-Wave Systems company was founded in 1999 with the aim of being the first private company in the world to create a useful quantum computer. After doing some research on core topics to decide which approach to use, they began to work with superconducting circuits, which we will explain in section 4.2, in 2004. Later, in 2007 they showed the first prototype of a quantum annealer called Orion. In 2010 they presented a paper showing a fully programmable block of 8 superconducting flux qubits in Ref.[3]. Using this block of 8 qubit as a basis, they constructed and sold to Lockheed Martin the first quantum computer named D-Wave 1 that had 128 flux qubits. The next year they announced the next model called D-Wave 2, which had the codename Vesuvius, that was launched in 2013. The machine I am going to analyze was presented in 2015. They have contracts with Google Quantum Artificial intelligence Laboratory(QuAIL), Nasa Ames laboratory, Los Alamos National Laboratory and the University of Southern California. In addition to everything mentioned so far, they offer the opportunity of testing the algorithms you have created for quantum annealers in the prototypes they have. The company has created a software[17] that enables the connection with the machine via internet.

[16] It will be explained in Section 4.2

[17] An interface using programming languages such as Python, C, ...

4.2 Technical aspects

4.2.1 Qubits

The qubit is the main concept in quantum computation as the bit is the most basic idea of the classical computation along with the Shannon entropy. On one hand, I am going to define the bit as the information we can get from a measure that has two posible outcomes e.g. flip a coin. In computer science, the possible outcomes of a bit are 1, e.g., if there is current, or 0, e.g., if there is not. On the other hand, the qubit is a two possible state quantum system such as a $s = \frac{1}{2}$ spin system, an atom that works with its ground state and the first excited state, a particle in a anharmonic potential in its ground and first excited state,... Any qubit can be represented with a quantum state denoted by this ket:[18]

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \ , \tag{36}$$

where $|0\rangle$ and $|1\rangle$ are the possible outcomes of a measurement, as in the classical bits, and $\alpha, \beta \in \mathbb{C}$ are complex numbers that obey the relation $|\alpha|^2 + |\beta|^2 = 1$.

As we can see in Eq.(36), a qubit can be in infinite possible states of superposition as far as it mantains the normalization condition. These states create a sphere, due to the normalization constraint, called Bloch sphere after F. Bloch, the first person who thought about it. However, the greatest majority of states are unaccesible to direct observations because when we measure the qubit it will always be in one of the classical states $|0\rangle$ or $|1\rangle$.

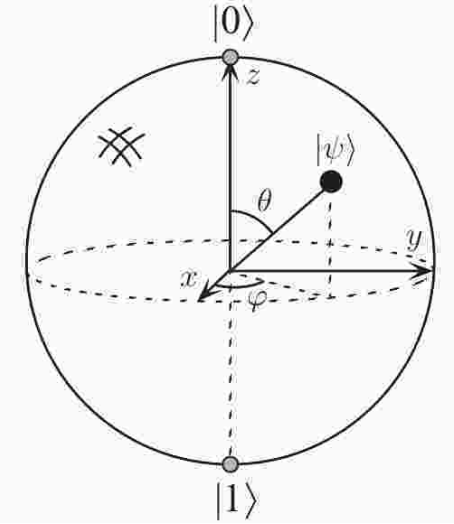

Figure 4: Bloch sphere. In this image we depict the states $|0\rangle$ as the north pole and $|1\rangle$ as the south pole of the sphere. Ref.:[25]

[18] Dirac notation will be used from this point.

4.2.2 Superconducting qubits

There are possible implementations for the qubits and its use in computation such as trapped ions[19], quantum dots[20],optical lattices[21], magnetic nuclear resonance quantum computers[22],etc . D-Wave $2X^{TM}$ like the other machines of the company employs a type of superconducting circuits called Superconducting Quantum Interference Devices(SQuID). The basic idea of the superconducting circuit can be understood watching Fig. 5.

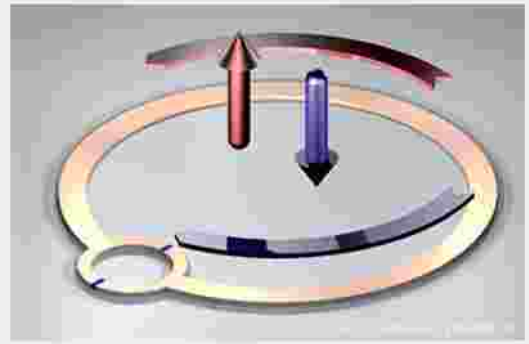

Figure 5: Idealization of a squid source. The lines in small circle represent the insulators. Ref.:[26]

The SQuID is made by a superconducting wire. In the case of D-Wave $2X^{TM}$, it is made of Niobium(Nb) whose critical temperature[23]. is 9.3 K, that has an insulator. For the D-Wave $2X^{TM}$ qubits they use AlO_x, that in principle opens the circuit. Moreover, this insulator is extremely thin and the Cooper pairs of the superconductors can tunnel through it to continue over the superconductor. This kind of structure is called Josephson junction, named after B. D. Josephson who predicted this phenomenon in 1962, and it enables currents in both ways creating magnetic fields that are superposed, what we were looking for.

This structure has a very big failure: the qubit response is too sensitive to the thickness of the junction and the fabrication errors can make the whole system completely useless. However, it can be corrected by considering two juctions in parallel. Then, two pairs are put in parallel as is clearly seen in Fig.6.

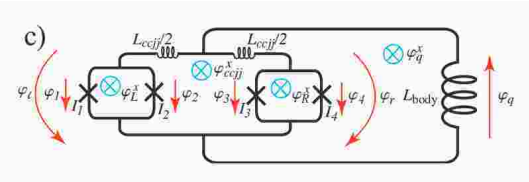

Figure 6: SQuID employed by the D-Wave machine. Ref.: [3]

[19]They consist in ions in a magnetic trap which are manipulated using lasers.

[20]Impurities of a solid state sample whose quantum state can be used as a qubit. The qubits can be manipulated using electrical currents.

[21]Neutral atoms that are trapped in optical cavities. The lattices can be manipulated using auxiliary lasers.

[22]They use a spin state of a molecule such as alanine that is manipulated using magnetic resonance techniques to simulate the qubit.

[23]The superconducting effects cease when the material is heated above the critical temperature

However, that is just a qubit and one cannot simulate the interactions among many of them with just a SQuID. In order to do that, the company employs 8 of them in a square array that can be seen in Fig.7. The interactions between the qubits are made by the couplers called CO and the transverse field is created by the RO. One can see the diagram of the 8 qubit architechture in Fig.8.

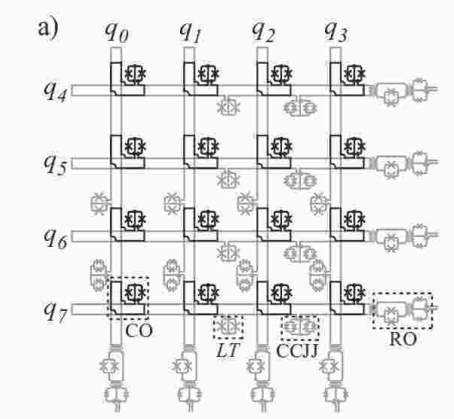

Figure 7: Arrangement of the 8 qubit block created by D-Wave. This is the basis of every chip created by the company. Ref.:[26]

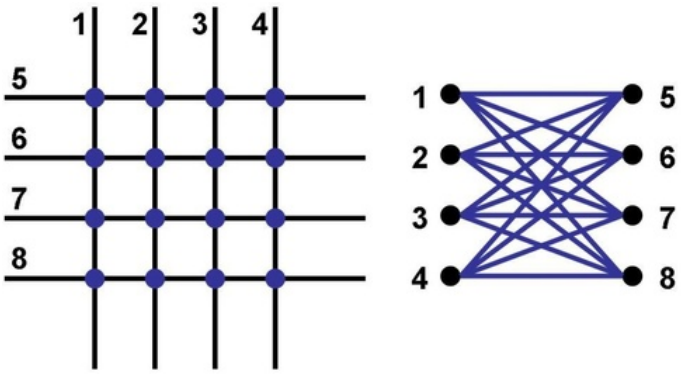

Figure 8: Graph of the 8 qubit block and the interactions among the qubits. Ref.:[27]

The 8 qubit block described in 4.2.2 can be scaled up to make chips of more qubits like the D-Wave 2^{TM} that has 2048 qubits. However, nowadays the qubits are not totally available because they must have interactions with their nearest neighbours. The architecture that can be seen for example in Fig. 9 is called Chimera. In this figure, we can appreciate the main difficulty of this architecture, that is to construct the maximum operative qubits ,i.e,. those which have connectivity with all its nearest neighbours letting the minimum number of qubits unoperative.

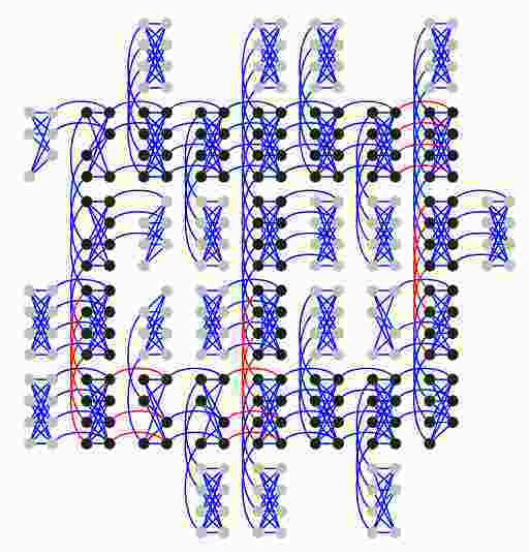

Figure 9: Graph of the Chimera architecture with 296 qubits. Ref.:[28]

4.2.3 Physical properties

[24] The chip that can be seen in Fig. 10 is kept at $15mK$, 180 times colder than the interstellar space, using a closed cycle dilution refrigerator system of multiple stages that can be appreciated in Fig. 11. This chip and its refrigeration system are contained in a $3m$ x $3m$ x $3m$ iron cube, see Fig. 12 that isolates the chip from the electromagnetic radiation. The magnetic field inside the cube is 50000 times smaller than the Earth magnetic field. Furthermore, the chip is in an extremely low pressure environment, the pressure is about 10^{-10} atm. Even though it may seem that the computer consumes much energy, it just consumes $25kW$, whereas a regular supercomputer consumes $4MW$. This can be a great advantage because according to D-Wave company, with some thousand qubits the power used will remain in the same order of magnitude.

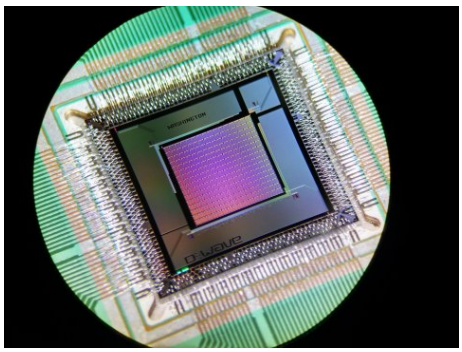

Figure 10: Photograph of the D-Wave 2^{TM} chip. Ref.:[26]

[24] All the information of this subsection is available in [26].

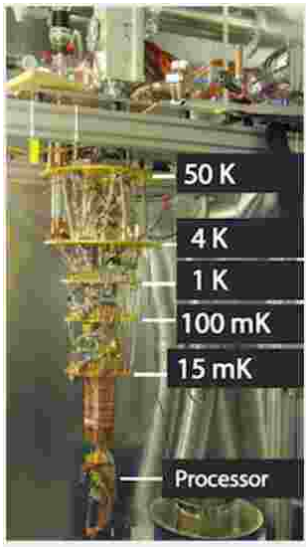

Figure 11: Refrigeration system. Ref.:[26]

Figure 12: Photograph of the D-Wave 2^{TM} protection cube. Ref.:[26]

4.3 Entanglement in D-Wave

4.3.1 Entanglement

Entanglement is a physical property that only appears in quantum systems, such that if it is detected in a system, it will be a key proof of the "quantumness" of its behaviour. The entanglement of a quantum system is related to the correlations of the measurements of certain observables of two or more particles of the system.

Suppose that we have two spin $s = \frac{1}{2}$ particles such as a particle-antiparticle pair that are created from the void. In the moment of the creation the particle-antiparticle pair one spin will be pointing up and the other one will be pointing down to conserve the total angular momentum, such their state will be the following,

$$|\psi\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \tag{37}$$

We will now separate the two particles far away and we will measure the spin of one particle. Then, we will always know the spin of the other particle will be the opposite of the one we have measured. This correlation can seem a violation of the speed of light as the maximum speed of information because these two particles can have an arbitrary distance between them. Nevertheless, this is a mistaken belief because you do not know the answer of the other experiment until you receive the result from the first experiment, which travels at most at the speed of light.

4.3.2 Measurements of entanglement in D-Wave 2^{TM}

The entanglement of the D-Wave $2X^{TM}$ machine has been demostrated in several experiments, see S. Boixo et al.[29], T. Lanting et al.[30] and S. Boixo et al.[31]. In the first article, they compare the results of the D-Wave 2 using 108 qubits, SA and SQA. They realize that there are strong correlations between the D-Wave 2 and the SQA results and weak correlations with the simulated annealing results which suggests that the D-Wave 2 is a quantum-mechanical device. In the second one, the group uses a technique called qubit tunneling spectroscopy which consists in using a probe qubit coupled to the system whose energy spectrum we want to measure. Then, the probe qubit state is measured until it tunnels incoherently from one state to another. This marks the energies of the system we want to measure. Using this technique it was demonstrated that the 8 qubit block tunnels coherently through the annealing proccess. Finally, in the third article they create a model of an open quantum system for the machine and compare the results of the model with the experimental ones.

4.4 Noise and error correction

4.4.1 Noise

The D-Wave $2X^{TM}$ as every other machine is affected by noise. In quantum computation as in classical computation, the effects of noise can be catastrophic and destroy the performance of the computer. Let us begin by defining the noise as an irregular random disturbance of the system that affects the correct performance of the computation. In our case, there are some disturbances such as the thermal fluctuations or electromagnetic radiation. Even though D-Wave 2^{TM} has the properties that I have described in section 4.2.3 that try to mitigate these effects, they are still very important and we have to consider them when we take into account the computational power of the chip.

These are just qualitative ideas about noise and to have a better understanding of it in the context of quantum systems I will briefly introduce the theory of decoherence. Quantum systems are not isolated and they interact with the enviroment, a system that is much bigger than the one we are analizing. Both systems create the so called universe whose evolution is unitary according to the following Hamiltonian:

$$H = H_e + H_s + H_{es} \ , \tag{38}$$

where H_e and H_s are the Hamiltonians of the enviroment and our system and H_{es} is the Hamiltonian of the interaction between system and enviroment.

As the evolution of the enviroment is irrelevant to us we can just focus on the evolution of the system and its interaction with the enviroment. However, we need a model to that interaction in order to make

computations. The easiest model has two approximations: it does not have memory (the interaction does not depend on the previous events) and the interaction can be taken as instantaneous(the interaction time is much smaller than the characteristic time of the system). From these suppositions, a master equation for the density matrix[25] is derived, the equation that explains the evolution of the system.

4.4.2 Error correction methods

As we have seen in Section 4.4.1 the noise disturbs the chip correct performance and we have to make some algorithm that prevents these errors or at least that corrects them when they occur. Some error correction codes have been created since the 1990's. Despite the majority of them are for digital implementations of quantum computation, see [25], the spectacular advance in analogic computation has created an interest in error correction algorithm for analogic devices such as quantum annealers. Moreover, some error correction algorithms for the D-Wave $2X^{TM}$ machine have been created. The first and most basic one has been created by Kristen L. Prudenz et al.[32]. It is based on two important changes,

- Reconfiguration of the chip qubits to make spontaneous qubit flips more unusual.
- Insert an extra term in the Hamiltonian, known as penalty term, to stabilize the state of the system.

On one hand, we have to distinguish two kinds of qubits in every 8 qubit block: the components of the logical qubits and the ancilla qubits. With 3 components we create a logical qubit that prevents spontaneous flips by locking ferromagnetically, where the coupling constant $J_{ij} = \beta$, the three components with the ancilla qubit. In Fig.13, the 4 qubits that create a logical qubit are represented by the same color where i_1, i_2, i_3 are the components and i_p is the ancilla term. Then, the Hamiltonian in Eq. (4) must be changed using the new variables $\bar{\sigma_i} \equiv \sum_{k=1}^{3} \sigma_{ik}^z$.

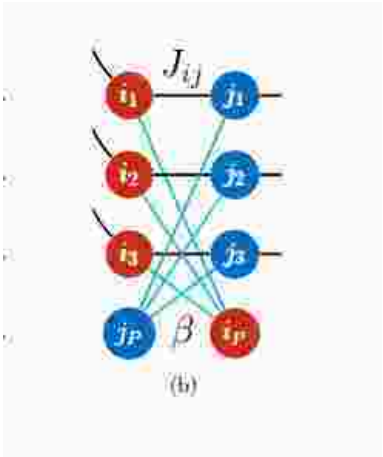

Figure 13: Scheme of the recofiguration of physical qubits to create logical qubits that can be protected from the enviroment. Ref.:[32]

The resulting Hamiltonian is,

[25] Matrix that represents the state of the quantum system. It can represent pure states as kets do or mixed states that are a statistical mixture of some pure states. $\rho \equiv \sum_i p_i |\psi_i\rangle \langle\psi_i|$ where $|\psi_i\rangle$ are kets of the pure states and p_i are the probabilities to be in the state $|\psi\rangle$.

$$\bar{H_{\text{Ising}}} = \sum_{i}^{\bar{N}} h_i \bar{\sigma_i^z} + \sum_{i<j}^{\bar{N}} J_{ij} \bar{\sigma_i^z} \bar{\sigma_j^z} \ , \tag{39}$$

where $\bar{\sigma_i^z}\bar{\sigma_j^z} \equiv \sum_{k=1}^{3} \sigma_{ik}^z \sigma_{jk}^z$, J_{ij} and h_i are the couplings between the logical qubits and the external field applied to the logical qubits.

On the other hand, the interaction of the components of the logical qubits with the ancilla qubit creates an extra term of the Hamiltonian that is given by,

$$H_P = \sum_{i}^{\bar{N}} \bar{\sigma_i^z} \sigma_{i_P}^z \ . \tag{40}$$

Then, the Hamiltonian of the machine using the protocol to protect it from the noise effects will be given by:

$$H_C = \alpha H_{\text{Ising}} - \beta H_P \ , \tag{41}$$

where $\beta \geq 0$ is the interaction described in the Fig 13, and $\alpha > 0$. Both constants have to be calibrated experimentally to achieve the best results.

4.5 New possible architecture of quantum annealers

The actual architecture of the D-Wave $2X^{TM}$ quantum annealer has a feature that its qubits just have interactions with their nearest neighbours. There are some optimization problems that need interactions beyond those. Last year, W. Lechner et al.[33] proposed an architecture that could simulate that. They codify N logical qubits in $K = \frac{N(N-1)}{2}$ physical qubits. However, this codification increases the degrees of freedom and we must apply $C_l = K - N + 1$ constraints, which are simulated using local interactions among the physical qubits. Then, the Hamiltonian of the problem will be,

$$H_p = \sum_{i}^{K} J_i \bar{\sigma_i^z} + \sum_{l}^{K-N+1} C_l \ , \tag{42}$$

where the vector J_i has all the independent components of matrix J_{jk}, which are local interactions among the physical qubits.
The constraints C_l are constructed as closed loops such that they must satisfy the following conditions: they must pass over all the physical qubits and be at least K-N. Furthermore, the states of the physical qubits represent the type of interaction among the logical qubits. If the spins of the logical qubit i and the logical qubit j are parallel the corresponding physical qubit will be in state 1. If they are not the physical qubit

will be 0. There are two posibilities of modeling the constraint, one using ancilla qubits see Eq.(43) and the other using a four body interaction see Eq.(44).

$$C_l = C(\sum_{m=\text{north,south,east,west}} \sigma^{z}_{(l,m)}) + S^z_l \tag{43}$$

$$C_l = C\sigma^{z}_{(l,n)}\sigma^{z}_{(l,s)}\sigma^{z}_{(l,e)}\sigma^{z}_{(l,w)} \tag{44}$$

Where north, south, east and west represent the qubits around each cross l. The geometrical arrangement that fits all these conditions is depicted in Fig 14.

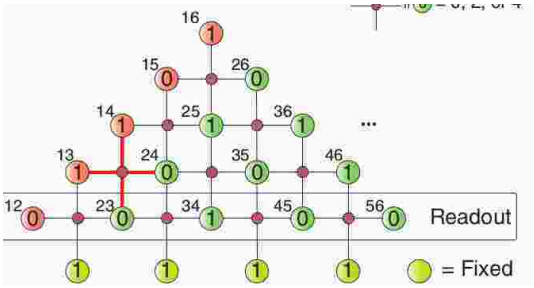

Figure 14: Scheme of the configuration used by W. Lechner et al.. Ref.:[33]

The scheme ilustrated in Fig. 14 is the following: the small red qubits are ancilla qubits that connect 4 physical qubits with local interactions. The first line of the scheme, called readout, will always be 1 in order to maintain all the scheme equal and simple. Thus, the scheme forces the number of spins that can be antiparallel, i.e., 0 in each cross to 0, 2 or 4. Then, the spins that are parallel, i.e., 1 have to be even.

4.6 Comparisons

Finally, I am going to describe the performance of the $D-Wave\ 2X^{TM}$ compared to different methods of optimization such as simulated annealing and simulated quantum annealing. Before we begin to compare them we have to define the gain of quantum annealing over the rest, also known as quantum speedup, as was introduced by T.F. Ronnow et al.[34],

$$S(N) = \frac{C(N)}{Q(N)} \ , \tag{45}$$

where $C(N)$ and $Q(N)$ are related to how do the classical and the quantum methods scale with the system size. Nevertheless, $C(N)$ and $Q(N)$ are not directly these scalings and we have to adjust them in order to make a good comparison. This happens because the two methods hardware does not have to scale equaly. Then, we have to adjust $C(N)$ and $Q(N)$ in the case of quantum annealing and simulated annealing. Therefore, $C(N) = \frac{T_{SA}}{N}$ and $Q(N) = T_{QA}$ such that the Eq.(45) now is,

$$S(N) = \frac{T_{SA}(N)}{T_{QA}(N)N} \ , \tag{46}$$

where $T_{SA}(N)$ and $T_{QA}(N)$ are the running times for the simulated and quantum annealing respectively.

D-Wave Systems company claimed that their machine runs 10^8 times faster than simulated annealing algorithms in [28]. However, they have not demonstrated yet that the quantum annealing method is faster than the fastest optimization algorithm. They predict that as the system size grows the quantum speedup will be a reality in some years.

5 Simulations

In order to show how the quantum annealing can beat the simulated annealing method I have made two simulations two find the minimum energy of a potential shown in Fig 15.

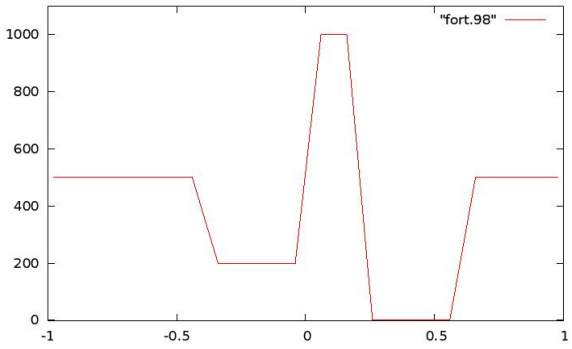

Figure 15: Potential that is resolved using simulated and quantum annealing

5.1 Simulated annealing

First of all, I have tried to solve the potential of Fig. 15 using simulated annealing as follows. I left a particle in a random place and let it evolve using thermal jumps using the algorithm I have explained in 3.2. The results for 100 random instances are shown in Fig. 16

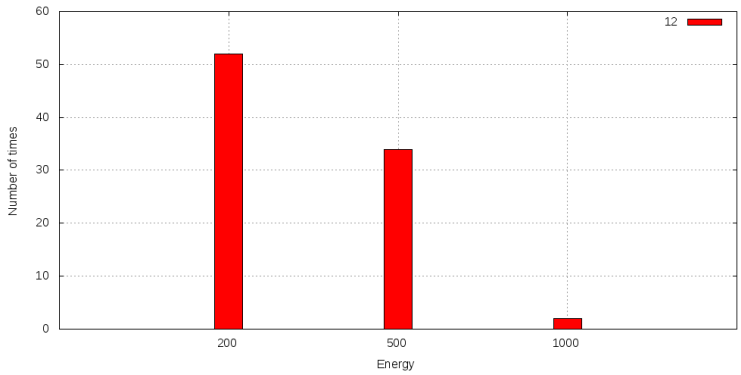

Figure 16: Results to 100 random instances. The global minimum results can be seen zoomed in in the box on the right. Half of the instances get stuck in the local minimum whereas just 12 arrive to the global minimum.

5.2 Quantum annealing

Next, I have used quantum annealing to solve the potential of Fig. 15. The system initial state is the ground state of an infinite potential well of lenght L. Then, the system slowly evolves, and the simulation of the time evolution has been made using the time-dependent Schrödinger equation with the Crank-Nicholson method, to the potential of Fig. 15. Finally, I show in Fig. 17 the ground state of the potential that I have obtained.

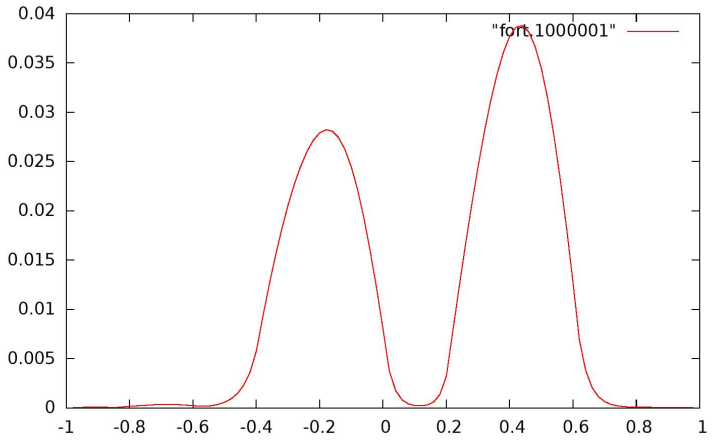

Figure 17: Ground state wavefunction using quantum annealing

6 Conclusions

All along this work I have reviewed the quantum annealing method, its use to solve combinatorial optimization problems and the best implementation nowadays, the D-Wave $2X^{TM}$ of the D-Wave Systems company. Firstly, in Section 2 I have introduced the most known and used combinatorial optimization problems such as traveling salesman problem (TSP) 2.1 and Boolean satisfability problem (k-SAT) 2.2.

Secondly, in Section 3 I have explained the algorithm that the D-Wave $2X^{TM}$ device employs. In subsection 3.1 I have explained the spin models of solids with special interest in Ising spin model. Then, in subsection 3.2 I have analized the simulated annealing method that was the inspiration of the quantum annealing. Next, I have studied the quantum annealing by describing the model it regularly uses in 3.3.1, explaining the quantum phenomena that makes quantum annealing a possibility, i.e., quantum tunneling in 3.3.2 and using a more general kind of computation called adiabatic quantum computation(AQC) to ensure the success of quantum annealing in 3.3.3. Later, in subsection 3.3.4 I have explained how can we solve the combinatorial optimization problems I have described in Section 2 using a quantum annealing approach and in subsection 3.3.5 I have explained two methods to simulate quantum annealing using computers.

Afterwards, in Section 4 I have explained quantum annealers in general and the D-Wave $2X^{TM}$ in particular. I have made a brief review of the history of the D-Wave Systems company and their quantun annealers in 4.1. Afterwards, I have analyzed the D-Wave $2X^{TM}$ chip taking care of its constituents, namely, the SQuID. This is why I have introduced the concept of the qubit and the inplementation that D-Wave employs, i.e., the superconducting circuits in 4.2. Later on, I have reviewed the entanglement of quantum systems in 4.3.1 and the proof that D-Wave $2X^{TM}$ works using quantum mechanics in 4.3.2. Subsequently, I have described how noise affects quantum systems 4.4.1 and an algorithm to prevent these errors in 4.4.2 . Afterwards, I have pointed out a new possible architecture that can simulate interactions betweeen any pair of qubits using just local Hamiltonians in 4.5. in 4.6 I have explained how the quantum annealers behave comparing with other algorithms such as simulated annealing or simulated quantum annealing.

Finally, in Section 5 I have made two simulations to solve the same potential, one using simulated annealing in 5.1 and the other using simulated quantum annealing in 5.2.

Quantum annealing has been showed to be an interesting method to solve combinatorial optimization problems that can possibly have significant speedup over the classical methods if it can correct the numerous limitations that it actually has. These include the effect of noise, the small quantity of qubits that are currently available and the limited connectivity among the qubits. The D-Wave Systems company has been a pioneer in the commercialization of quantum computer prototypes. Despite the prohibitive price for particulars of their current devices they may be an affordable purchase for universities and companies all around the globe.

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