Quantum Memristors

Author: Paul Pfeiffer

Supervisors:
Prof. Enrique Solano
Prof. Jan von Delft

Master Program
Theoretical and Mathematical Physics

Ludwig-Maximilians-Universität München
Technische Universität München

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Abstract

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by Paul Pfeiffer

We introduce quantum memristors which consist in quantum systems subject to a decoherence mechanism that is controlled by the history of the quantum state. These systems constitute the quantum analogue of classical electrical circuits involving memristors, that is, resistors whose resistance depends on the history of the voltage across them. We propose a master equation capturing the dynamics of quantum memristors on the basis of a measurement-based feedback scheme. Our findings show that the memory effect can persist in a quantum memristor, but suggest that recovering its maximum value goes along with classicalisation. Finally, we discuss further research lines involving quantum memristors in current quantum technologies.
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Chapter 1

Introduction

Motivation

Memristors are two-terminal passive electrical circuit elements with a resistance value that depends on the history of the voltage signal across or the current signal through them. These devices have the capability of memory in the sense of providing a response to an external signal depending on the previous stream of inputs. Their dynamical reaction grants them a special position among the standard set of passive circuit elements. They were the essential building block in a circuit model of neuron membrane [1], appeared as the missing fourth element in circuit theory [2] and are again in the focus of investigation since their recent experimental demonstration by Hewlett-Packard [3].

The memory features of the memristor are different from pure storage. Information encoded in an electrical signal is not only registered by the memristor but processed at the same time. The field of memristive computing is exploring the power of this local combination of storage and processing in performing computation [4]. From an abstraction of the memristor, the concept of a memory element has been proposed that serves as the building block of a new computing architecture, called the universal memcomputing machine [5].

The field of quantum technologies may profit from the memristor due to its attractive features as a circuit element and as a computational unit. The theory of circuit quantum electrodynamics describes the quantum nature of electrical signals in superconducting lumped circuits [6], a prominent platform of quantum technologies. It is known that one of the key element of circuit QED, the Josephson junction, has a memristive component [7]. Can the memory features of the memristor be advantageous in the quantum realm? Especially its computational power poses the question, whether it could be exploited in quantum simulations or quantum information processing. More generally,
regarding the memristor as an example of beneficial memory effects, how to improve quantum technologies using memory capabilities?

Scope of this thesis

In this thesis, we introduce quantum memristors, a class of open quantum systems with decoherence depending on the history of a system variable. They represent a conceptual quantum analogue of classical electrical circuits involving a memristor. Proposing a way to model the dynamics of quantum memristors, we are able to test whether the memory features are altered and how they affect the quantum system.

Due to its conceptual nature, this thesis does not touch the mesoscopic physics that would govern the behaviour of a concrete circuit setup involving a real memristor. It is rather aimed at shining light on the peculiar nature of information extraction and feedback in a quantum system that come along with the memory capability of a memristor. A caveat is in order, this work is not related to the field of quantum memory which is concerned with the mere storage of quantum information. In particular, in our model the information about the history of the system variable is stored in a classical variable. However, due to the history-dependent decoherence, quantum memristors have a natural connection to the concept of non-Markovianity and can be related to quantum non-Markovian processes, the class of linear quantum dynamics with memory [8].

Previous work along the line of memory elements in quantised electrical circuits includes the study by Di Ventra of the non-dissipative relatives of the memristor, namely the meminductor and the memcapacitor [9]. Their Lagrangian approach works for the quantisation of those energy conserving elements, but cannot deal with the dissipation induced by the memristor. Furthermore, Yurke and Denker quantise the RSJ-model of a Josephson junction [10], including the memristive contribution, but this does not allow a direct generalisation for an arbitrary memristor.

Approach to quantum memristors and results

In order to handle the history dependent decoherence, we adopt a view of a memristor that identifies its three main traits: dissipation, information extraction and conditioned action. We regard a memristor as a tuneable linear resistor operated by a classical control unit that acts conditioned on a continuous stream of voltage measurements. The control unit chooses the resistance value according to the past measurement records. In this way, the aggregate of tuneable resistor and control unit reproduces the history dependent response of a memristor.
Along these lines, we propose a way to incorporate the three memristor traits into the
dynamics of a quantum system. We follow Devoret in modelling the effect of a linear
resistor in a quantised electrical circuit via a linear, weak coupling to a harmonic oscillator
bath [11]. The controllability of the resistor is added by making the damping rate in
the Ohmic spectral density of the bath a tuneable parameter. This corresponds to a
controlled Hamiltonian of circuit and bath. To complete the above scheme, this control
has to be conditioned on the results of a continuous monitoring of one circuit variable.
Due to the active nature of quantum measurements, this monitoring also contributes to
the dynamics. Hence, we propose to describe the history dependent decoherence as a
result of a measurement-based feedback scheme on the level of circuit and bath [12].

Our main result consists in a quantum memristor master equation based on these grounds.
In a numerical case study of Gaussian states in an LC-circuit coupled to a memristor,
we provide the first example of quantum memristor dynamics. Taking hysteresis as the
mark for memory, we find that in general the hysteresis is less significant due to the
nature of the quantum measurement process. However, if intrinsic measurement noise
and back action are sufficiently suppressed, the memory effect in form of the hysteresis
has the same emphasis as in the classical case. Finally, we critically discuss this regime
with respect to the quantum-classical transition, because continuous measurements are
candidates to provide classical dynamics in the form of well localised-trajectories [13].

Overview of the chapters

Chapter 2 prepares the three pillars that this work is built on. First, the quantisation of
the electrical signals in a lumped circuit is presented and the Devoret model of a linear
resistance in a quantum electrical circuits is introduced [10, 11]. A brief summary of the
range of general quantum dynamics, including open quantum systems, quantum measure-
ment, feedback control and quantum Markovianity, starts the second part. In particular,
a review of the Nakajima-Zwanzig projector technique is taken as a starting point for
a discussion the Born-Markov approximation and a derivation of the Caldeira-Leggett
master equation [14]. Then the stochastic master equation for continuous measurements
is shown, the difference between conditioned and unconditioned quantum evolution is ex-
plained and a short summary of measurement-based feedback is given [12, 15]. The last
part of chapter 2 deals with the theory of memristors. It includes the general description
of memory circuit elements, provides an illustration of their computational power and
reviews the Lagrange approach to their quantisation [5, 9].

Chapter 3 presents a central result of the thesis, the quantum memristor master equation.
It starts with a detailed discussion of the feedback model of a memristor. On these
grounds a phenomenologically motivated form of the quantum memristor dynamics is
proposed. Its non-linear, history dependent form is commented on with respect to the concept of quantum non-Markovianity. This chapter concludes with a road map for a rigorous microscopic derivation of the quantum memristor master equation.

Chapter 4 shows a numerical study of the quantum memristor master equation for the case of Gaussian states in a LC circuit coupled to a memristor. The measurement strength is identified as the primary parameter that determines the survival of the memory. An analysis of the respective contributions of measurement noise and back action in terms of this parameter allows to identify a regime where the hysteresis in the unconditioned evolution recovers the classical level. Finally, the hypothesis that the memory effect comes along with its classicalisation is debated.

Chapter 5 concludes the thesis. It evaluates, how the idea of the quantum memristor has and might further contribute to the original question about gains in quantum technologies using beneficial memory effects.
Chapter 2

Background & Basic Concepts

The purpose of this thesis is to study systems subject to dissipative dynamics with memory in the quantum regime. A natural platform for this study are electrical circuits. One the one hand, the field of circuit quantum electrodynamics has characterised theoretically and verified experimentally the quantum nature of electrical signals in superconducting circuits. On the other hand, there is revival of interest in the memristor, a resistor whose resistance depends on the voltage history across the element.

This chapter prepares the discussion of joining these two worlds and introduces the notions used in the subsequent chapters. Hence, in the first part the quantisation of electrical circuits is reviewed. A short discussion of two celebrated quantum circuit representatives, the LC resonator and the Cooper-pair box, shows the necessity to be able to incorporate dissipation in the formalism. This part concludes with the Devoret model of a linear dissipative circuit element via an infinite ensemble of LC resonators. The second part puts this model in the frame of open quantum systems and general quantum dynamics beyond unitary dynamics. The Caldeira-Leggett master equation as a prime example of dissipation in quantum systems is reviewed. Further, the dynamics introduced by continuously monitoring a system is shown to be captured by the stochastic master equation, which is also the basis to consider the powerful control tool of feedback. In the last part, memory circuit elements are introduced and the idea behind their novel information processing capabilities is explained. Finally, the last section examines a Lagrange approach to quantise circuits with memory elements and points out the problems of incorporating the memristor in this frame.
2.1 Quantised electrical signals in superconducting circuits

The quantum nature of electrical signals is an example of macroscopic quantum mechanics. Although on a microscopic level an electric circuit is a complicated condensed matter system of $10^{23}$ particles, under certain conditions the collective coordinates of charge and flux can be treated as continuous quantum variables [10]. In this regime, a superconducting coil printed onto a silicon chip, is described by the physics of an inductor. Hence, the description of an electric circuit in a Hamiltonian formalism contains parameters like inductances or capacitances. Different from the case of quantum systems like single atoms, these parameters are not constants of nature, but can be engineered [11]. This high degree of configurability is an important feature for quantum technologies.

2.1.1 Lagrangian and Hamiltonian formalism for circuits

The most common technique to quantise a physical system is the construction of the Lagrangian of the system and imposing commutation relations on its set of conjugates variables. Thus, the first step in constructing a quantisation scheme for electrical circuits is the identification of its degrees of freedom, i.e. a choice of canonical coordinates. This review will follow the prescription presented by Devoret [11].

In the lumped element model, an electrical circuit is described by a set of nodes connected by directed branches which consist of the various circuit elements. The voltages over these elements and the currents flowing through them determine the state of the circuit. The voltage over a branch $b$ is defined by the integral of the electric field connecting the two corresponding start and end nodes $n_b, n'_b$,

$$V_b(t) = \int_{n_b}^{n'_b} E \, dl,$$

while the current is given by integrating the magnetic field on a loop $\Gamma_b$ around the branch

$$I_b(t) = \frac{1}{\mu_0} \int_{\Gamma_b} B \, dl.$$

The set of branch variables includes furthermore the branch flux\(^1\)

$$\Phi_b(t) = \int_{-\infty}^{t} V_b(s) ds$$

\(^1\)These are definitions valid for all two leads electrical elements. They generalise the physical notions of magnetic flux and electrical charge for circuit elements where a physical correspondence, like flux through a coil or charge on a capacitor, lacks.
and the branch charge

\[ Q_b(t) = \int_{-\infty}^{t} I_b(s) ds. \tag{2.4} \]

The branch variables over-determine the state of the circuit because they are not independent quantities. They are restricted by the Kirchhoff laws representing energy and charge conservation. The first law states that magnetic fluxes in branches which form a closed loop sum to the flux through this loop. Hence, every loop \( l \) in the network, possibly enclosing an external magnetic flux \( \Phi_{ext}^l \), sets a restriction

\[ \sum_{b \in l} \Phi_b = \Phi_{ext}^l. \tag{2.5} \]

The second law relates the branch charges connected to the same node, requiring for every node \( n \), possibly charged with external charge \( Q_{ext}^n \),

\[ \sum_{\{b|n \in b\}} Q_b = Q_{ext}^n. \tag{2.6} \]

The first Kirchhoff law provides a convenient way of constructing a set of independent variables. Every loop in the circuit reduces the degrees of freedom of the circuit by one. Removing branches of the circuit until it is loop free, while keeping all nodes connected, leaves behind a spanning tree of the original graph. In a tree, two nodes \( n, n' \) are connected by a unique path \( P(n, n') \). Therefore, choosing a node as the ground node \( n_g \) allows to define a node flux for all the remaining active nodes by

\[ \varphi_n = \sum_{b \in P(n_g, n)} D(P(n_g, n), b) \Phi_b, \tag{2.7} \]

where \( D(P, b) \) is \( \pm 1 \) if \( b \) goes along (against) the direction of the path. The node fluxes together with the restrictions of the first Kirchhoff law determine all the branch fluxes. Node fluxes are not unique, because they depend on the choice of spanning tree and ground node.

A set of node fluxes represents all degrees of freedom of the circuit. A Lagrangian of the node fluxes and their derivatives gives the equations of motion, which are obtained by the second Kirchhoff law combined with the constitutive current voltage relation of the circuit elements. For capacitive elements, the corresponding branch charge and branch voltage are related by

\[ V_b = c_b(Q_b). \tag{2.8} \]
In general, $c_b(Q_b)$ is a non-linear function, but for a capacitor it is linear, $c_b(Q_b) = \frac{Q_b}{C_b}$.

For inductive elements, a relation for branch flux and branch current is valid

$$I_b = b_b(\Phi_b),$$

(2.9)

where two cases of $b_b(\Phi_b)$ are of special interest, the linear inductor with $b_b(\Phi_b) = L_b\Phi_b$ and the non-linear Josephson element, $b_b(\Phi_b) = I_c \sin\left(2\pi \frac{\Phi_b}{\Phi_0}\right)$; see Section 2.1.1.2.

Replacing the branch fluxes via the node fluxes and using the restrictions of the second law the equations of motion are obtained. For an example see the treatment of the LC circuit in section 2.1.1.1. The current through a capacitive element is proportional to the second derivative of the flux, whereas the current through inductive elements is a function of the flux. Therefore, the equations of motion have a structure that allows to identify them with the Euler-Lagrange equations of a Lagrangian

$$\mathcal{L}^{\prime}(\{\phi_n\}, \{\dot{\phi}_n\}; \{Q_{ext}\}, \{\Phi_{ext}\}); n \in \{\text{Active nodes}\}.$$

The conjugate momenta to the node fluxes are defined by

$$q_n = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_n}$$

(2.10)

and are equal to the total amount of charges on the capacitors connected to the corresponding node. The conjugate momenta are well-defined quantities, since parasitic capacitances guarantee that every node is connected to a capacitor.

The Legendre transform provides the Hamiltonian of the circuit

$$H(\{\phi_n\}, \{q_n\}; \{Q_{ext}\}, \{\Phi_{ext}\}).$$

The final step towards the quantum treatment of the circuit consists in treating the canonical coordinates as operators (denoted by hats) with the canonical commutation relations

$$[\hat{q}_n, \hat{q}_{n'}] = [\hat{\dot{\phi}}_n, \hat{\dot{\phi}}_{n'}] = 0, [\hat{\phi}_n, \hat{q}_{n'}] = i\hbar \delta_{n,n'}.$$  

(2.11)

2.1.1.1 Quantisation of the LC circuit

The LC circuit has a capacitive branch with $Q_C = CV_C$ and an inductive branch with $I_L = \frac{\Phi_L}{L}$. If the inductive branch is chosen as the spanning tree (see Figure 2.1), the node flux $\phi$ equals the flux of the inductive branch, $\phi = \Phi_L$. With the orientation of the
branches depicted in the figure, the Kirchhoff laws Equation 2.5 and Equation 2.6 read

\[ \Phi_L - \Phi_C = 0, \]
\[ Q_L + Q_C = 0. \]

The first equation determines the two branch fluxes via the node flux, \( \varphi = \Phi_L = \Phi_C \). The second equation becomes the equation of motion for the flux by deriving it with respect to time and inserting the respective expressions for the branch currents. It reads

\[ C\ddot{\varphi} = -\frac{\varphi}{L}, \quad (2.12) \]

which is the Euler-Lagrange equation associated with

\[ \mathcal{L}(\varphi, \dot{\varphi}) = \frac{1}{2}C\dot{\varphi}^2 - \frac{1}{2}L\varphi^2. \quad (2.13) \]

Thus, the conjugate momentum is

\[ q = C\dot{\varphi} \quad (2.14) \]

and corresponds to the charge on the capacitor, which can be seen by rewriting the time derivative of the flux as \( \dot{\varphi} = \dot{\Phi}_C = V_C \).

The corresponding Hamiltonian reads

\[ H(\varphi, q) = \frac{q^2}{2C} + \frac{\varphi^2}{2L}. \quad (2.15) \]

It is the sum of the electrostatic energy in the capacitor and the energy stored in the inductor due to the magnetic flux. Its form suggests a mechanical analogue of the LC circuit. Flux can be regarded as position and charge as momentum of a particle of mass.
C attached to a spring of strength $\frac{1}{k}$ [11, p. 366]. Hence, the quantum LC circuit, where flux and charge become operators respecting the commutation relations, represents an example of the well-known quantum harmonic oscillator.

Some basic results of the quantum harmonic oscillator which are necessary to follow the discussion of the quantisation of circuits involving linear resistances in Section 2.1.2 are stated here. The frame of quantum mechanics is introduced later in Section 2.2. It is convenient to rewrite the flux and charge operators in terms of the ladder operators

$$\hat{\phi} = \sqrt{\frac{\hbar}{2C\omega}}(a + a^\dagger),$$
$$\hat{q} = \frac{1}{i}\sqrt{\frac{\hbar C\omega}{2}}(a - a^\dagger),$$

where $\omega = \frac{1}{\sqrt{LC}}$ is the circuit frequency.

The corresponding canonical commutation relation for the ladder operators is

$$[a, a^\dagger] = 1.$$ (2.18)

In terms of these operators, the Hamiltonian takes its familiar form

$$\hat{H} = \hbar\omega (a^\dagger a + \frac{1}{2}).$$ (2.19)

It is a diagonal representation in the number basis, the eigenbasis of the number operator $\hat{n} = a^\dagger a$. The eigenvalues of $\hat{n}$ are positive integers which equal the number of energy quanta in the circuit.

The time dependence of the ladder operators is given by

$$a(t) = ae^{-i\omega t},$$ (2.20)
$$a^\dagger(t) = a^\dagger e^{i\omega t}.$$ (2.21)

Furthermore, in the case of a thermal state at inverse temperature $\beta = \frac{1}{k_B T}$ their second order expectation values read

$$\langle a^\dagger a \rangle = \frac{1}{2} \left[ \coth \left( \frac{\beta \hbar \omega}{2} \right) - 1 \right],$$
$$\langle a a^\dagger \rangle = \langle a^\dagger a \rangle + 1, \langle a(t)^\dagger a(t) \rangle = 0.$$ (2.22)

The LC resonator is a crucial element in circuit QED, because it provides the circuit analogue of a cavity in quantum optics. One condition to observe the quantised nature of the electrical signals in an LC circuit is the suppression of thermal excitations. This
requires the level spacing to be large in comparison with the thermal energy and the level broadening due to inevitable coupling to uncontrollable degrees of freedom. These parasitic degrees of freedom can be of very different nature, but their primary origin is the coupling to external circuitry and defects in the superconducting leads. The effect of this coupling may be summarised in an effective admittance, usually assumed to be Ohmic, $Y(\omega) = R^{-1}$, in parallel to the LC circuit [16]. The corresponding level broadening is given by the relaxation time of the circuit $\tau = \frac{1}{RC}$. The conditions for observing the discrete energy levels are low temperatures $k_B T \ll \hbar \omega$ and a high quality factor, $Q = \frac{\omega}{\tau} = \frac{R}{Z} \gg 1$, where $Z = \sqrt{LC}$ is the resonators impedance.

A further condition to study non-classical regimes of the resonator is the ability to generate and measure genuine quantum states like Fock states or highly squeezed states [17]. Naturally occurring states, like coherent or thermal states, are in a correspondence limit and show a behaviour that only in the higher moments differs from the classical one. Also exciting the resonator by external driving is not able to go beyond these states. The key to access the resonators genuine quantum nature is coupling it to a non-linear circuit element [18]. Furthermore, introducing a non-linear circuit element generates spectra with non-uniform level spacing. In this case, a qubit, an excitation restricted to a two-dimensional subspace of the Hilbert space, can be studied and manipulated.

### 2.1.1.2 Josephson junctions

Josephson junctions (JJ) are an important building block in superconducting circuits for many reasons, among them their non-linear inductance, which led to multiple circuit proposals to build superconducting qubits [19]. A rigorous treatment of the JJ needs a discussion of superconductivity. But in a semi-classical picture, the JJ can be modelled by a classical circuit and can be discussed in the frame of the quantisation formalism of circuits introduced above. Especially interesting is the fact that the dissipative elements in the circuit model of the JJ comprise a memristor. Hence, the JJ is the natural example, where both, the memristor and the quantisation of circuits appear.

A Josephson junction consists of a thin isolating interface between two superconductors. The interface builds a capacitance and due to its small width Cooper pairs can tunnel across the junction, which leads to the Josephson current. The Josephson current depends on the phase difference $\delta$ of the order parameters of the superconductors across the junction [20],

$$I_S(t) = I_c \sin(\delta(t)),$$  \hspace{1cm} (2.23)
where $I_c$ is the critical current. If there is a voltage bias $V(t)$ across the junction, the phase difference evolves according to

$$\dot{\delta}(t) = \frac{2\pi}{\Phi_0} V(t),$$

with the superconducting flux quantum $\Phi_0 = \frac{h}{2e}$. Hence, the phase difference is proportional to the branch flux $\Phi_S$ across the junction\(^2\) and the Josephson current can be understood as the response of a non-linear inductor, the Josephson element, with current-flux relation [16]

$$I_S(t) = I_c \sin \left(2\pi \frac{\Phi_S(t)}{\Phi_0} \right).$$

The current-voltage relation of an ideal JJ, which includes the capacitance, but works dissipation free, is thus mimicked by a Josephson element in parallel with a capacitor. If the ideal JJ is placed between a grounded superconductor a small isolated superconducting island, this builds the setup for the Cooper Pair Box (CPB), see Figure 2.2. In the same way that the LC circuit was treated, the quantised circuit Hamiltonian of the CPB is obtained,

$$\hat{H} = \frac{\hat{q}^2}{2C} - \frac{\Phi_0 I_c}{2\pi} \cos \left(2\pi \frac{\hat{\phi}}{\Phi_0} \right).$$

Due to the cosine term in the Hamiltonian, the energy spectra of the CPB is not equally spaced and allows to implement a qubit [21].

\(^2\)Because $\delta(t)$ is a phase difference, it is only defined modulo $2\pi$. This also implies that all observable quantities depending on the phase difference are $2\pi$-periodic functions and in this sense it can always be replaced by the branch flux.
The JJ is not only subject to dissipative process due to external circuitry, but has inbuilt dissipative elements. The Cooper pairs are energetically protected by the superconducting gap, but they can break up due to temperature or a bias voltage. Breaking up cooper pairs generates quasi-particles in the superconducting condensate, essentially electrons. The tunnelling of these quasi-particles is dissipative.

The most simplified circuit model of a dissipative junction is the resistively shunted junction (RSJ). In this circuit representation of the JJ, the additional dissipative current contribution is realised by shunting the ideal JJ by a Ohmic resistor. However, since the original paper of Josephson, it is known that there is a further dissipative current which depends on the phase difference $\delta(t)$. The dissipative current in an extension of the RSJ model, see Figure 2.3, comprising the phase dependent conductance has the form [7, ch. 4],

$$I_D(t) = G [1 + \epsilon \cos (\delta(t))] V(t),$$  \hspace{1cm} (2.27)

where $G$ is a conductance and $\epsilon$ is a real number with $|\epsilon| \leq 1$. In a recent paper, Di Ventra emphasises that the current-voltage relation of the phase dependent conductance coincides with the response of a voltage-controlled memristor [22]. Because the phase difference equals the time integral over the previous voltage amplitudes, the conductance depends on the history of the voltage. This is the mark of a memristor, see Section 2.3.

### 2.1.2 Devoret model of a linear resistance

The quantum realm of the dissipation free electrical circuits can be studied in the framework of the Hamiltonian formalism, because it presents a straightforward way of quantisation. Dissipation in electrical circuits is ubiquitous due to coupling to external control and read-out circuits or the quasi-particle currents in the Josephson junction. Still, its effects in a quantum system cannot be studied in a Hamiltonian framework. The reason is the time reversal symmetry of the Hamilton equations in a closed system which needs
to be broken to incorporate dissipation, a process with a preferred time direction [11, p. 366].

An instructive way to reconcile dissipation with time-reversibility, and thus with a Hamiltonian description, is recalling that dissipation describes the process of converting energy in the system of interest into heat. But heat is essentially energy in a huge number of uncontrollable degrees of freedom i.e. a further set of physical systems. If these environmental degrees of freedom are included in the model, a closed system is obtained that can be studied in a Hamiltonian framework. Finally, integrating out these additional degrees of freedom, one might study their influence, and hence the influence of dissipation, on the subsystem of interest.

Realistic modelling of the environmental degrees of freedom is a very hard task, because the microscopic origins of dissipation are numerous and difficult to characterise. But there is a universal criterion for a microscopic model of a dissipative process. It should reproduce the fluctuation-dissipation theorem that relates the amount of fluctuations in a system to the strength of the dissipative process [23, p. 162]. Therefore, it serves as a guide in designing a model for an environment.

2.1.2.1 Quantum fluctuation-dissipation theorem in circuits

The fluctuation-dissipation theorem in electrical circuits relates the dissipation of a linear circuit element to the Nyquist noise, namely the fluctuating currents and voltages across it. Devoret proposed a circuit representation of a linear dissipative circuit element that reproduces the Nyquist noise using only capacitors and inductors [11]. As the circuit only contains non-dissipative elements, it can be quantised in the Hamiltonian framework. Hence, it yields a quantum version of the Nyquist noise and allows to study the effect of a linear resistance in a quantum circuit. Devoret’s network of capacitors and inductors is the circuit version of the Caldeira-Leggett model of a bath of harmonic oscillators, which constitutes a general tool in modelling environments [24].

Every linear circuit element is characterised by its admittance $\tilde{Y}(t)$. The admittance determines the current response in the case of an applied voltage signal $V(t)$,

$$I(t) = \int_{-\infty}^{\infty} dt' \tilde{Y}(t') V(t - t'). \quad (2.28)$$

Causality sets the constraint $\tilde{Y}(t) = 0$ for $t > 0$. Decomposing the applied voltage signal into its frequency components, one can also regard the current response in each frequency regime,

$$I(\omega) = Y(\omega)V(\omega). \quad (2.29)$$
The admittance in frequency space is obtained via the Fourier transform of the time response function,

\[ Y(\omega) = \int_{-\infty}^{\infty} dt \tilde{Y}(t)e^{i\omega t}. \] (2.30)

In the case of an applied current, the voltage response is given by the impedance \( Z(\omega) \) which is the inverse of the admittance.

A positive real part of the response function of a linear circuit element leads to dissipation and in accordance with the fluctuation-dissipation theorem determines the amplitude of the current and voltage noise across it. If a dissipative circuit element is connected to an amperemeter or a voltmeter, the current, respectively voltage, fluctuates. These noisy currents and voltages have a zero mean value, but a finite variance, which is related to their spectral density,

\[ C_I(\omega) = \int_{-\infty}^{\infty} dt \langle I(t)I(0) \rangle e^{i\omega t}, \] (2.31)
\[ C_V(\omega) = \int_{-\infty}^{\infty} dt \langle V(t)V(0) \rangle e^{i\omega t}. \] (2.32)

According to the Nyquist theorem, temperature and the real part of the admittance or the impedance give the current, respectively voltage, spectral density [23, p. 19], [11, p. 370],

\[ C_I(\omega) = 2k_B T \Re Y(\omega), \] (2.33)
\[ C_V(\omega) = 2k_B T \Re Z(\omega). \] (2.34)

Therefore, a circuit model of linear dissipative circuit element has to provide a real part in the admittance and it has to reproduce the current and voltage fluctuations given by the Nyquist theorem. Devoret shows that an infinite number of parallel series LC circuits provides these two features.

First, let us consider a series LC circuit of capacitance \( C_j \), inductance \( L_j \) and circuit frequency \( \omega_j = \frac{1}{\sqrt{L_j C_j}} \). Despite the purely imaginary admittance of the capacitor and the inductor,

\[ Y_L(\omega) = -\frac{1}{i\omega L_j}, \] (2.35)
\[ Y_C(\omega) = -i\omega C_j, \] (2.36)
the series LC circuit has a real part in the admittance. Its origin are the two singularities in the admittance of the LC circuit \(^3\),

\[
Y_j(\omega) = \left( \frac{1}{Y_{C_j}(\omega)} + \frac{1}{Y_{L_j}(\omega)} \right)^{-1} = \frac{i}{2L_j} \left( \frac{1}{\omega - \omega_j} + \frac{1}{\omega_j + \omega} \right).
\]

(2.37)

The transformation back to the time domain,

\[
\hat{Y}(t) = \int_{-\infty}^{\infty} Y(\omega) e^{-i\omega t},
\]

(2.38)

shows that the causality constraint requires \(Y_j(\omega)\) to be an analytic function in the lower half plane. In this case, Jordan’s Lemma guarantees that \(\hat{Y}_j(t > 0) = 0\) [25, p. 718]. In order to provide the analyticity of \(Y_j(\omega)\), the singularities in Equation 2.37 are pushed slightly into the upper half plane. The admittance is a response function, which means that it is always tested against some function to obtain an observable quantity. Hence, it might be regarded as a distribution and using the distribution identity, \(\frac{1}{x + i0} = \text{P.V.} \left( \frac{1}{x} \right) + i\pi\delta(x)\), its real part is apparent [11, p. 368]

\[
Y_j(\omega) = \frac{i}{2L_j} \left( \frac{1}{\omega - \omega_j - i0} + \frac{1}{\omega + \omega_j - i0} \right) = \frac{1}{Z_j} \left( \frac{\pi}{2} \omega_j \left[ \delta(\omega - \omega_j) + \delta(\omega + \omega_j) \right] - \frac{i}{2} \left[ \text{P.V.} \left( \frac{1}{\omega_j - \omega} \right) + \text{P.V.} \left( \frac{1}{\omega_j + \omega} \right) \right] \right).
\]

(2.39)

A parallel ensemble of series LC circuits like in Figure 2.4 has an admittance whose real part is a collection of delta peaks,

\[
\Re Y(\omega) = \sum_j \Re Y_j(\omega).
\]

(2.40)

Hence, an infinite ensemble of LC oscillators with incremental frequency spacing and properly chosen inductances and capacitances mimics the real part of any given admittance function \(Y(\omega)\). For a thorough proof, see [26, p. 4420]).

In order to prove the validity of the ensemble of LC circuits as a model for a dissipative element, it remains to show that the corresponding current and voltage fluctuations obey the Nyquist theorem. A quantum treatment of the circuit provide these thermal

\(^3\)The admittance for two linear circuit elements \(Y_1, Y_2\) in parallel is the sum of the individual admittances \(Y_{||} = Y_1 + Y_2\). In series the total admittance is given by the inverse of the sum of the reciprocal admittances, \(Y_{ser} = \left( Y_1^{-1} + Y_2^{-1} \right)^{-1}\).
fluctuations in the classical limit and reveals the form of the quantum noise in the non-classical regime.

Using the rules from Section 2.1.1 the Hamiltonian of the circuit in Figure 2.4 has the form

\[ H = \frac{q^2}{2C} + \frac{\varphi^2}{2L} + \sum_j \frac{q_j^2}{2C_j} + \frac{(\varphi_i - \varphi)^2}{2L_i}. \]  

(2.41)

The first two terms correspond to the oscillator, which is used as a test object for the influence of the ensemble of LC oscillators that mimics the admittance. But before, the current fluctuations across the LC circuits are derived in the absence of this test oscillator. In this case, the Hamiltonian is a collection of independent harmonic oscillators.

In contact with a heat bath at temperature \( T \), all the LC circuits are in a thermal state and the charge on their capacitors fluctuates around a zero mean. The correlation function of the total charge \( Q = \sum_j q_j \) with respect to the thermal states yields

\[
\langle Q(t)Q(0) \rangle = \sum_j \langle q_j(t)q_j(0) \rangle \\
= \sum_j \frac{\hbar}{2Z_j} \left( \langle a_j a_j^\dagger \rangle e^{-i\omega_j t} + \langle a_j^\dagger a_j \rangle e^{i\omega_j t} \right) \\
= \sum_j \frac{\hbar}{2Z_j} \int_{-\infty}^{\infty} d\omega \frac{\omega}{\omega} (\delta(\omega - \omega_j) + \delta(\omega + \omega_j)) \left[ \coth \left( \frac{\beta\hbar\omega}{2} \right) + 1 \right] e^{-i\omega t} \\
= \sum_j \frac{\hbar}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{1}{\omega} \left[ \coth \left( \frac{\beta\hbar\omega}{2} \right) + 1 \right] \Re Y_j(\omega) e^{-i\omega t}. \]

(2.42)

\[ ^4 \text{From this point on, the hats on the operators are omitted.} \]
Here, the second moments of the ladder operators from Equation 2.22 have been used and the delta functions were replaced by the real part of the admittance of the \( j \)-th oscillator. The sum over the oscillators allows express the charge-charge correlation in terms of the admittance \( Y(\omega) \) of the circuit element that the LC chain was designed to mimic. The spectral density of the charge fluctuations is the Fourier transformation of the time correlation function and can be read directly from Equation 2.42,

\[
C_Q(\omega) = \frac{\hbar}{\omega} \left[ \coth \left( \frac{\beta \hbar \omega}{2} \right) + 1 \right] \Re Y(\omega). \tag{2.43}
\]

Finally, the charge correlation function is related to the current correlation function by

\[
\langle I(t)I(0) \rangle = \langle \dot{Q}(t)\dot{Q}(0) \rangle = -\frac{d^2}{dt^2} \langle Q(t)Q(0) \rangle,
\]

which for the spectral density results in

\[
C_I(\omega) = \omega^2 C_Q(\omega) = \hbar \omega \left[ \coth \left( \frac{\beta \hbar \omega}{2} \right) + 1 \right] \Re Y(\omega). \tag{2.44}
\]

Likewise, the spectral density of voltage fluctuations across a linear dissipative circuit element in terms of its real part of the impedance \( \Re Z(\omega) \) has the form [11, p. 374]

\[
C_U(\omega) = \frac{\hbar}{\omega} \left[ \coth \left( \frac{\beta \hbar \omega}{2} \right) + 1 \right] \Re Z(\omega). \tag{2.45}
\]

These two relations represent the circuit version of the general quantum fluctuation-dissipation theorem by Kubo [27].

For the classical part of the spectrum \( \beta \hbar \omega_{cl} \ll 1 \) i.e. these frequencies where the LC circuits are highly populated and the fluctuations are primarily thermal, the Nyquist noise is reproduced,

\[
C_I(\omega_{cl}) \approx 2k_B T \Re Y(\omega_{cl}). \tag{2.46}
\]

Moreover, fluctuations with frequencies \( \beta \hbar \omega_q \gg 1 \) far from the classical range are not due to thermal noise. The LC oscillators in this frequency range are in the ground state. They cannot emit any excitations and the spectral density vanishes for the negative frequencies in the quantum range. For positive, it reflects the fact that only full quanta of energy \( \hbar \omega \) can be absorbed

\[
C_I(\omega_q) = 2\hbar \omega_q \Re Y(\omega_q). \tag{2.47}
\]
2.1.2.2 Quantum ohmic resistor

The ensemble of quantum LC resonators fulfills the requirement of reproducing the fluctuation-dissipation theorem. Consequently, the effect of a linear Ohmic resistor coupled to the LC circuit in Figure 2.4 can be studied using the Hamiltonian Equation 2.41. For this purpose, the capacitance and inductance values of the LC ensemble have to be chosen according to an Ohmic admittance,

\[ Y(\omega) = R^{-1} \frac{1}{1 - i \frac{\omega}{\omega_c}}, \]  

with resistance \( R \) and the cut-off frequency \( \omega_c \gg \omega \).

Describing the dynamics of the LC circuit is a problem of the field of open quantum systems, which is addressed in Section 2.2. For high temperatures, this leads to the Caldeira-Leggett master equation, which is shown in Section 2.2.1.2.

Devoret is only interested in the fluctuations of the circuit variables in thermal equilibrium. Using the quantum fluctuation-dissipation theorem, he confirms the expected thermal broadening in the high temperature limit. However, for very low temperatures of the bath, he finds that the flux variance is suppressed with respect to the isolated circuit. In return, to preserve the Heisenberg uncertainty principle the charge variance grows [11, p.377].

![Figure 2.5: The coupling to the linear resistor decreases the flux variance for small temperature and shows the expected linear increase for larger temperatures. The resistance values are parametrised by \( \gamma = \frac{1}{\sqrt{\epsilon_0 C \omega}} \). For the plot, the expression for the flux variance [11, Eq. (3.36)] has been used with \( \phi_0^2 = \frac{A}{\pi \epsilon_0 C \omega} \) and \( \theta = \frac{k_B T}{\hbar \omega} \).](image-url)
2.2 Open quantum systems, measurement and feedback

In the language of non-relativistic quantum mechanics, the fundamental physics are unitary evolutions of pure states complemented by von Neumann measurements. This bottom layer of quantum physics constitutes the field of isolated quantum systems, whose evolution, generated by a constant Hamiltonian, is free of decoherence and memoryless. On this level, the contact of quantum systems with the external world is only contained in the measurement formalism.

However, accessible to observation are small parts of globally isolated systems, called open quantum systems. The surrounding of these observable subsystems is called environment. Due to the interaction with the environment an open quantum system can be subject to a rich variety of evolutions, which build the second layer of quantum dynamics. In particular, the missing information about the state of the environment allows for memory effects and leads to decoherence.

In the following, a brief introduction to open quantum systems, general measurements and feedback is given. It is based on [14, 28] and connects closely to the three succeeding subsections: the Caldeira-Leggett master equation, the stochastic master equation to describe continuous measurement and the discussion of quantum feedback. These tools constitute the frame for the introduction of the quantum memristor master equation in chapter 3.

In the first subsection, the Nakajima-Zwanzig projection operator technique is presented as a way to establish the link between the layer of isolated and the layer of open quantum systems. In general, the evolutions obtained via this link are linear completely positive maps for the density operator. Moreover, the structure of these mappings is used to distinguish Markovian, forgetful, and non-Markovian, reminiscent, time evolution. The first subsection concludes presenting the Caldeira-Leggett master equation.

In the second subsection, a description of a continuous measurement process by the stochastic master equation is provided. The collapse and renormalisation of a quantum state after a von Neumann measurement make quantum measurements non-linear, dynamical processes. This departure from a linear evolution is discussed, because measurements are the basis for feedback mechanisms.

In the third subsection, the implementation of feedback via the Wiseman-Milburn scheme is reviewed. At first sight, feedback lies outside of the two layers of quantum dynamics, since it requires a classical observer, who performs measurements and conditioned control on the quantum system. However, a formulation of feedback involving a probe system
and conditioned unitaries places it among the evolution of open quantum systems in the second layer.

The density operator

The state of a physical system is characterised by the measurement statistics an observer can obtain from it. Therefore, a description always depends on the observer’s state-of-knowledge e.g. insights into the system preparation. The object that contains all the measurement statistics is the density operator. It is a bounded, linear operator on a $D$-dimensional Hilbert space\(^5\),

$$
\rho \in \mathcal{B}(\mathcal{H}),
$$

that is positive and has trace norm 1. This requirements allow to express it in its spectral decomposition\(^6\),

$$
\rho = \sum_j \lambda_j \ketbra{j}{j} \text{ with } \sum_j \lambda_j = 1 \text{ and } \forall j : \lambda_j \geq 0. \quad (2.49)
$$

The $\lambda_j$ play the role of a probability, denoting the occurrence of the state $\ket{j} \bra{j}$ in the ensemble defined by $\rho$.

In the bottom layer of isolated quantum systems, the evolution of a state $\rho$ and the prescription to extract measurement statistics are given in the form of two postulates:

**Isolated systems evolve unitarily with a time-independent generator**

An isolated quantum system evolves unitarily according to the Liouville-von Neumann equation,

$$
\frac{d}{dt} \rho = -i [H, \rho], \quad (2.50)
$$

whose solution for an initial state $\rho_0$ reads

$$
\rho(t) = U(t)\rho U(t)^\dagger \quad (2.51)
$$

with $U(t) = \exp (-iHt)$. Trace preservation requires the generator of the evolution, the Hamiltonian $H$, to be a Hermitian operator.

**Von Neumann measurements**

Every choice of a normalised basis \{\ket{m}\} defines a measurement via the set of projectors $P_m = \ketbra{m}{m}$. The measurement device selects the state $\ket{k}$ with probability,

$$
p_k = \text{Tr} (P_k \rho). \quad (2.52)
$$

---

\(^5\)In this introductory discussion only finite dimensional Hilbert spaces are considered.  
\(^6\)The sum over the index $j$ is always to be understood as $\sum_{j=1}^{D}$. 

The state after the measurement of $k$ is

$$\rho_k = \frac{P_k \rho P_k}{\text{Tr}(P_k \rho P_k)}. \quad (2.53)$$

An observable $A$ is a Hermitian operator and its expectation value is measured by performing a von Neumann measurement according to its eigenbasis $\{|a\rangle\}$ and weighing each eigenvalue $\lambda_a$ with the corresponding occurrence probability $p_a$,

$$\langle A \rangle = \text{Tr} A \rho. \quad (2.54)$$

The link of these two postulates to the second layer of quantum systems in contact with an environment consists in two mathematical tools: the description of composite systems by the tensor product and the representation of measurement statistics of subsystems by the reduced density matrix. The state of a composite system consisting of two parts, e.g. two particles, assigned to Hilbert spaces $\mathcal{H}_P$ and $\mathcal{H}_Q^7$, is a density operator on the Hilbert space,

$$\mathcal{H} = \mathcal{H}_P \otimes \mathcal{H}_Q.$$

The second postulate implies that the measurement statistics of an observer who has only access to part $P$ of the composite state $\rho \in \mathcal{B}(\mathcal{H}_P \otimes \mathcal{H}_Q)$, i.e. his observables have the form $A_P = A \otimes 1$, are contained in the reduced density operator $\rho_P$[29, p. 107]. The reduced density operator in $P$ is obtained via the partial trace of the joint density operator $\rho$ over the other subsystem $Q$,

$$\rho_P = \text{Tr}_Q \rho. \quad (2.55)$$

**Completely positive maps**

The first postulate and the partial trace characterise the form of quantum dynamics of subsystems. Every unitary $U$ acting on the joint system can be recast choosing any basis $\{|q_j\rangle\}$ of the second subsystem [28, p. 23],

$$U = \sum_{j,j'} A_{jj'} \otimes |q_j\rangle \langle q_{j'}| \quad (2.56)$$

with $A_{jj'} = \langle q_j | U | q_{j'} \rangle \Rightarrow \forall j \sum_j A^*_{jj'} A_{jj'} = 1. \quad (2.57)$

---

7In general, they can be of different dimensions, but this is not explicitly treated in the following
Hence, a composite system initially in a product state of the form $\rho = \rho_P \otimes |q_0\rangle \langle q_0|$ evolves according to Equation 2.51 into

$$\rho(t) = U(t)\rho U(t)^\dagger = \sum_j A_{0j}(t)\rho_P A_{0j}^\dagger(t) \otimes |q_j\rangle \langle q_j|.$$ (2.58)

Denoting $A_{0j}(t) = A_j(t)$ and performing the partial trace over $Q$, the evolved state of subsystem $P$ reads

$$\rho_P(t) = \sum_j A_j \rho_P A_j^\dagger =: \Phi(t)\rho_P.$$ (2.59)

Due to the condition Equation 2.57 the reduced state $\rho_P(t)$ fulfils the requirements of a density operator i.e. gives a valid physical state. The completeness of the link between the first and the second layer is provided by the important result of Choi that every valid time evolution has the form of $\Phi(t)$, i.e. can be written in a operator-sum operation with condition Equation 2.57 [30]. A valid time evolution maps density operators to density operators and fulfils complete positivity [28, p. 27].

The quantum evolution $\Phi(t)$ is parametrised by time. It is called completely positive (CP)-divisible [31, p. 2] iff

$$\forall s \in (0, t) \exists \Phi(t, s) \text{ completely positive map: } \Phi(t) = \Phi(t, s)\Phi(s).$$ (2.60)

If additionally $\Phi(t, s) = \Phi(t - s)$, the one parameter family $\Phi(t)$ forms a semigroup [14, p. 119].

Another mean to express the evolution of the subsystem $P$ is a differential equation for $\rho_P(t)$. Its exact form is obtained by taking the partial trace over $Q$ of the Liouville-von Neumann equation,

$$\frac{d}{dt} \rho_P = \text{Tr}_Q \left( -i \left[ H, U(t)\rho_P \otimes |q_0\rangle \langle q_0| U(t)^\dagger \right] \right).$$ (2.61)

A systematic way to approximate this expression via a closed equation for $\rho_P$, called a quantum master equation, is discussed in Section 2.2.1. Often these approximations allow to cast the master equation in a Lindblad form [14, p. 123],

$$\frac{d}{dt} \rho = -i[H, \rho] + \sum_k \gamma_k \left( L_k \rho L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho - \frac{1}{2} \rho L_k^\dagger L_k \right).$$ (2.62)

The Lindblad operators $L_k$ are arbitrary and the damping coefficients $\gamma_k$ have to be positive. The Lindblad equation presents the most general form of a master equation, whose
solution has the semigroup property. If the damping coefficients are time-dependent, positive functions, the semigroup property of the solution reduces to CP-divisibility.

**Quantum non-Markovianity**

The field of quantum non-Markovianity is a very active research field and in a recent review the authors cautiously warn that the field is not 'at a stage that allows us to decide on a definitive definition of quantum Markovian processes' [8, p. 3]. Their review builds the basis of the summary below.

Markovianity is a property that was first defined for classical stochastic processes and characterises memoryless processes. It is defined via conditional probabilities, stating that in a Markovian process the conditional probability of a random variable $X$ to have value $x$ at time $t$, given it took the values $y_N, \ldots, y_1$ at times $t > t_N > \cdots > t_1$, depends only on the most recent pair $y_N, t_N$. Hence, the Markovian process "forgot" about all the previous values of $X$.

Regarding the density operator as an extension of a probability distribution, a direct translation of this classical definition is problematic due to the active perturbing nature of quantum measurements. The measurement process chosen to obtain a conditional probability alters the quantum dynamics, which one wants to test for Markovianity [8, p. 10]. Hence, a quantum evolution could only be classified to be Markovian with respect to a certain measurement scheme.

Most measurement-independent definitions of quantum Markovianity rely on the fact that in the classical case, Markovianity is equivalent to a divisibility property of the transition matrix that maps the initial probability distribution to the probability distribution at a later point in time. Adopting this fact to quantum evolutions $\Phi(t)$, a quantum evolution is defined to form a quantum Markovian process iff it is CP-divisible, i.e. fulfilling Equation 2.60. Hence, every master equation that cannot be cast into a time-dependent Lindblad form lead to a quantum non-Markovian evolution. Obviously, unitary evolutions quantum Markovian processes.

However, a master equation can take multiple forms and thus usually the mathematical properties of its solution are tested to detect the violation of CP-divisibility i.e. the presence of non-Markovianity. In [32], the authors propose to detect non-Markovianity in single time slide by regarding the distance of the quantum evolution at a time $t$ to all the maps that are solutions of time-dependent Lindblad equations. Although this detects all non-Markovian processes, from the calculational point of view, a more tractable witness of non-Markovianity was introduced by Breuer [33]. It uses the contractive property of completely positive maps. Consider a distance measure in the space of density operators,
like the trace distance,
\[ D(\rho_1, \rho_2) := \frac{1}{2} \text{Tr} |\rho_1 - \rho_2|. \] (2.63)

After the states have been evolving according to the completely positive map \( \Phi(t) \), the distance contracts
\[ D(\Phi(t)\rho_1, \Phi(t)\rho_2) \leq D(\rho_1, \rho_2). \] (2.64)

If \( \Phi(t) \) is CP-divisible, then the distance of any two initial states monotonically decreases, because for \( s < t \) it holds
\[ D(\Phi(t)\rho_1, \Phi(t)\rho_2) = D(\Phi(t, s)\Phi(s)\rho_1, \Phi(t, s)\Phi(s)\rho_2) \leq D(\Phi(s)\rho_1, \Phi(s)\rho_2). \] (2.65)

The states become less and less distinguishable, because information is lost to the environment. A back-flow of information is only possible for a non-Markovian evolution, marked by a temporary increase in the state distance. Hence, integrating the violations of this monotonic decrease during the evolution and maximising this number over all initial states, gives a measure of non-Markovianity,
\[ N(\Phi) = \max_{\rho_1, \rho_2} \int dt \frac{d}{dt} D(\Phi(t)\rho_1, \Phi(t)\rho_2). \] (2.66)

In [8, p. 21], the authors remark that \( N(\Phi) > 0 \) provides only a sufficient condition for quantum non-Markovianity. Accordingly, there are non-Markovian evolutions yielding \( N(\Phi) = 0 \), although they are not CP-divisible.

**General measurements and non-linear quantum dynamics**

Composite quantum systems also allow to extend the set of possible measurements. In the case of two interacting systems \( P \) and \( Q \), the second system \( Q \) can also be regarded as a probe to extract information about the state of \( P \). The probe is prepared in a known state, brought close to the system of interest, during the interaction correlations between system and probe build up and afterwards a von Neumann measurement according to the basis \( \{|q\rangle_j\} \) is preformed [28, p. 24]. The measurement device of the probe \( Q \) displays the result \( k \) with probability,
\[ p_k = \text{Tr} \left( A_{k}^{\dagger} \rho_{P} A_{k} \right), \] (2.67)

and the final state of the system \( P \) is
\[ \rho_{P}^{(k)} = \frac{A_{k}^{\dagger} \rho_{P} A_{k}}{p_k}. \] (2.68)

Different from all the dynamics described by linear completely positive maps, this change from the initial to the final state is a non-linear function of the density operator. Its
non-linearity stems from the selection of a subspace via the von Neumann measurement and the rescaling that is necessary to preserve the trace of the post-measurement density operator. The non-linear dynamics of monitored quantum systems lead to non-linear master equations, see Section 2.2.2.

**Measurement-based feedback and coherent feedback**

If the observer of a quantum system is equipped with control tools allowing to shape the Hamiltonian of the system, she can implement feedback protocols of the following form. The observer performs a general measurement with outcomes labelled by $k$. In the case of the outcome $k$, he controls the system such that it evolves according to the unitary $U_k$. The conditioned final state is thus obtained by evolving the post-measurement state in Equation 2.68 according to the unitary $U_k$,

$$
\rho_P^{(k)} = \frac{U_k A_k \rho P A_k^\dagger U_k^\dagger}{\text{Tr} \left( A_k \rho P A_k^\dagger \right)}.
$$

(2.69)

Due to the measurement process, this constitutes a non-linear evolution. The infinitesimal time limit for feedback based on continuous measurement is sketched in Section 2.2.3 and leads to a corresponding non-linear master equation for the conditioned state. However, in most cases, the density operator of interest is given by the ensemble of the conditioned states, because the measurement result is not controllable. The unconditioned state reads

$$
\rho_{fb} = \sum_k p_k \rho_P^{(k)} = \sum_k U_k A_k \rho P A_k^\dagger U_k^\dagger.
$$

(2.70)

The unconditioned evolution is a linear evolution and can be implemented without performing any measurement [28, p. 236]. After the interaction of probe and system to prepare the measurement, the observer sets aside the von Neumann measurement and instead fixes a joint evolution of probe and system according to a unitary of the form,

$$
U_{fb} = \sum_k U_k \otimes |q_k \rangle \langle q_k|.
$$

(2.71)

It is easy to see that in this measurement-free protocol, the reduced density operator of the system $P$ has the same form as the ensemble of conditioned states in Equation 2.70. Hence, based on this small argument it is expected that feedback protocols always lead to a linear evolution for the unconditioned state.
2.2.1 Dynamics of open quantum systems

The evolution of open quantum systems differs from a unitary evolution due to the coupling to the environment. These dynamics are in generally very involved and require approximations to be solved. The Nakajima-Zwanzig projection technique methodologically investigates the mutual influence of system and environment to provide a general structure of the system evolution. In particular, this structure is especially suited to discuss the commonly used Born-Markov approximation, which is the starting point to derive the Caldeira-Leggett master equation for a system coupled to the canonical bath of harmonic oscillators. The summary here is based on [14, ch. 9.1]

The basic idea of the projection operator technique is a systematic split of a global state of system and environment via super operators into a system-relevant and a system-irrelevant part and to study the coupled form of their time evolution.

The state of the composite system (s) and environment (e) is a density operator in the composite Hilbert space,

\[ \rho \in \mathcal{B}(\mathcal{H}_s \otimes \mathcal{H}_e), \]  

whose dynamics is generated by a Hamiltonian,

\[ H = H_s + H_e + \alpha H_I, \]  

such that its time evolution in the interaction picture (superscript \( (D) \)) reads

\[ \dot{\rho}^{(D)}(t) = \mathcal{L}(t)\rho^{(D)}(t) = -i\alpha \left[ H_I^{(D)}(t), \rho_I(t) \right]. \]  

The super operator \( \mathcal{L}(t) \) represents the commutator and the operators in the interaction frame are the Schrödinger operators evolved with respect to the dynamics of isolated system and environment,

\[ A^{(D)}(t) = \exp \left( i(H_s + H_e) t \right) A \exp \left( -i(H_s + H_e) t \right). \]  

In the following the superscript is dropped for convenience.

However, all information accessible within the system is contained in the reduced density operator given by the partial trace over the environmental states,

\[ \rho_s(t) = \text{Tr}_e \rho(t). \]
A convenient method to study the time evolution of this reduced density operator consists in decomposing the global state in the relevant part,

\[ P_\rho := \text{Tr}_e [\rho(t)] \otimes \rho_B, \]  
(2.77)

and the irrelevant part,

\[ Q_\rho := \rho - P_\rho, \]  
(2.78)

where \( \rho_B \) is a fixed, normalised state of the environment. Together the two super-operators recover the whole state

\[ P + Q = 1 \]  
(2.79)

where 1 is the identity. These act like projections and the image of one is in the kernel of the other,

\[ P^2 = P, \]  
(2.80)

\[ Q^2 = Q, \]  
(2.81)

\[ PQ = QP = 0. \]  
(2.82)

These relations allow to rewrite the time evolution of the global state as a coupling between the relevant and the irrelevant part,

\[ \frac{d}{dt} P_\rho(t) = P\mathcal{L}(t)\rho(t) = P\mathcal{L}(t)P_\rho(t) + P\mathcal{L}(t)Q_\rho(t), \]  
(2.83)

\[ \frac{d}{dt} Q_\rho(t) = Q\mathcal{L}(t)\rho(t) = Q\mathcal{L}(t)P_\rho(t) + Q\mathcal{L}(t)Q_\rho(t). \]  
(2.84)

The formal solution of the second equation reads,

\[ Q_\rho(t) = G(t,t_0)Q_\rho(t_0) + \int_{t_0}^{t} ds \, G(t,s)Q\mathcal{L}(s)P_\rho(s), \]  
(2.85)

where

\[ G(t,s) = T_{\leftarrow} \exp \int_s^t Q\mathcal{L}(s')ds'. \]  
(2.86)

with the time ordering operator \( T_{\leftarrow} \).

Plugging this solution back into the equation for the relevant part gives a closed equation in \( P_\rho \) called the Nakajima-Zwanzig (N-Z) equation,

\[ \frac{d}{dt} P_\rho(t) = P\mathcal{L}(t)P_\rho(t) + G(t,t_0)Q_\rho(t_0) + \int_{t_0}^{t} ds \, P\mathcal{L}(t)G(t,s)Q\mathcal{L}(s)P_\rho(s). \]  
(2.87)
This equation is exact, but in general is very hard to tackle. Usual assumptions to simplify its structure are first an initial product state,

$$\rho(t_0) = \rho_s(t_0) \otimes \rho_B,$$

which in an experimental setup corresponds to an initial preparation of the system state that destroys all entanglement to its surroundings. Second, the environmental state $\rho_B$ is assumed to have a structure which makes the odd moments of the interaction Hamiltonian vanish,

$$\text{Tr}_e(H_I(t_1)\ldots H_I(t_{2k+1})\rho_B).$$

(2.89)

A typical example is an environment of harmonic oscillators in the Gibbs state and a linear interaction Hamiltonian. The equilibrium state is diagonal in the number basis and a linear interaction is proportional to the ladder operators $a$ or $a^\dagger$ whose odd moments clearly vanish.

A factorising initial condition implies that the second term in the N-Z equation drops,

$$Q\rho(t_0) = 0,$$

(2.90)

and the vanishing odd moments make the first term zero,

$$PL(t)P\rho(t) = \text{Tr}_e(-i[H_I,\text{Tr}_e(\rho(t)) \otimes \rho_B]) \otimes \rho_B = 0.$$  

(2.91)

The remaining term of the evolution contains the essential effect of the interaction between system and environment. Part of the information contained in the system state is transferred to the environment and may at a later point in time affect the evolution. The resulting memory effect is represented by the time integral over the history of the state,

$$\frac{d}{dt} P\rho(t) = \int_{t_0}^{t} ds \ K(t, s) P\rho(s),$$

(2.92)

where the memory kernel super operator,

$$K(t, s) = PL(t)G(t, s)QL(s),$$

(2.93)

determines the importance of the state at time $s$ for the update at time $t$. 

2.2.1.1 Born-Markov approximation

The main characteristics of the memory kernel, namely amplitude and temporal width, depend on the strength of the interaction, the internal dynamics of the environment and its size. Amplitude and width are the starting points for approximations to simplify the N-Z equation.

The Born approximation

By introducing a dimensionless interaction strength parameter, $H_I \rightarrow \alpha H_I$, the memory kernel can be written in a perturbative expansion that starts at second order in $\alpha$,

$$K(t, s) = \alpha^2 P \mathcal{L}(t) Q \mathcal{L}(s) + \mathcal{O}(\alpha^3).$$  \hspace{1cm} (2.94)

The second order could reflect that the memory is a two-fold process, involving the interaction twice, in transferring system information to the environment and in its partial back flow. In this weak interaction approximation, the N-Z equation reads,

$$\frac{d}{dt} \rho(t) = \alpha^2 \int_{t_0}^{t} dr \left( P \mathcal{L}(t) \mathcal{L}(r) P \rho(r) + P \mathcal{L}(t) P \mathcal{L}(r) P \rho(r) \right)$$
$$= -\alpha^2 \int_{t_0}^{t} dr \text{Tr}_e ([H_I(t), [H_I(r), \rho_s(r)] \otimes \rho_B]) \otimes \rho_B,$$ \hspace{1cm} (2.95)

for which $Q = 1 - P$, the simplification due to Equation 2.91 and the identification of the system state in Equation 2.76 have been used.

Markov approximation

The second characteristic of the memory kernel is its temporal width, which determines the time during which information having been transferred to the environment is still available and can flow back to the system. Hence, the temporal width shrinks with the size of the environment, because the information rapidly spreads over the huge number of degrees of freedom and cannot be recovered in any finite time. Environments of this size are called baths. The time scale of the information degradation in the bath is the correlation time $\tau_B$, i.e. the time during which a correlation of two environmental operators survives,

$$\langle B_i(t + \tau_B) B_j(t) \rangle \neq 0.$$  \hspace{1cm} (2.96)

Assuming that the dynamics of the system happens at a much slower time scale as $\tau_B$, the narrow memory kernel allows to replace $\rho_s(r)$ by $\rho_s(t)$ in eq. Equation 2.95. Here only the system dynamics due to the interaction namely the dissipation and decoherence time scales are of importance, because $\rho(t)$ is in the interaction picture. Finally, after
tracing out the environment, a time-local equation for the system density operator is obtained,

$$\frac{d}{dt} \rho_s(t) = -\alpha^2 \int_{t_0}^{t} dr \, \text{Tr}_e \left[ [H_I(t), [H_I(r), \rho_s(t) \otimes \rho_B]] \right].$$  \hspace{1cm} (2.97)

In this expression, called the Redfield master equation, the memory effect is squeezed into a quasi instantaneous back kick on the system. Still, the evolution in this form is not memoryless, since it makes a difference for the update at distinct times due to the integration range. But this effect should be negligible as the support of the memory kernel is concentrated close to $t$. Hence, the integral can be approximated by replacing the lower integration boundary, $t_0 \to \infty$, and after a change of the integration variable, $s \to s - t$, the Redfield master equation turns into the memoryless Born-Markov master equation,

$$\frac{d}{dt} \rho_s(t) = -\alpha^2 \int_{0}^{\infty} dr \, \text{Tr}_e \left[ [H_I(t), [H_I(t-r), \rho_S(t) \otimes \rho_B]] \right].$$  \hspace{1cm} (2.98)

The transformation back to the Schrödinger picture gives

$$\frac{d}{dt} \rho^{(s)}_S(t) = -i \left[ H_s, \rho^{(s)}_s(t) \right] - \alpha^2 \int_{0}^{\infty} d\tau \, \text{Tr}_e \left[ [H_I, [H_I(-\tau), \rho^{(s)}_s(t) \otimes \rho_B]] \right].$$  \hspace{1cm} (2.99)

### 2.2.1.2 Caldeira-Leggett master equation

A very general environment is the bath of harmonic oscillators: an ensemble of non-interacting bosonic modes that are weakly linearly coupled to the system of interest. It has been extensively studied to understand decoherence and is a commonly used tool to model dissipation [34]. The evolution of the system can be studied via various techniques, e.g. the Feynman-Vernon influence functional [35], the quantum Langevin equation in a Heisenberg picture approach [26] or the Caldeira-Leggett master equation [36].

The following section focuses on the latter, because master equations provide the most convenient language to study continuous measurement and feedback. The electrical circuit version of a harmonic oscillator bath was introduced in Section 2.1.2 and it allowed to discuss the equilibrium fluctuations of current and voltage in a quantum circuit coupled to a linear resistance. The Caldeira-Leggett master equation describes the time evolution of the circuit towards this equilibrium in the high-temperature limit. The following overview uses the language of quantum circuits in order to present a coherent picture.

First, a review of the bath correlation functions clarifies the Born-Markov approximation of the last section and introduces the spectral density which summarises all the relevant physical properties of the bath. Second, the high-temperature limit is discussed that, in connection with a slow system evolution, allows to further simplify the Born-Markov
master equation and cast it into the Caldeira-Leggett form. Finally, the induced decoherence and dissipation is analysed and compared to the classical equations of motion for a circuit coupled to a resistance. This section is based on [14, ch. 3.6]

The Hamiltonian of a circuit coupled to an ensemble of series LC circuits in Equation 2.41 splits into a circuit, a bath and an interaction term,

\[
H = H_{\text{circ}} + \sum_k \frac{q_k^2}{2C_k} + \frac{(\varphi - \varphi_k)^2}{2L_k} \\
= H_{\text{circ}} + \sum_k \frac{\varphi^2}{2L_k} - H_I + H_B
\]  

with the interaction, \( H_I = \varphi B \), for which the bath operator \( B \) has been defined as

\[
B = \sum_k \frac{1}{L_k} \sqrt{\frac{\hbar}{2C_k \omega_k}} \left( b_k + b_k^\dagger \right),
\]

and the bath Hamiltonian,

\[
H_B = \sum_k \hbar \omega_k b_k^\dagger b_k,
\]

where \( b_k \) is the annihilation operator for the mode with frequency \( \omega_k \) with the canonical commutation relations. The circuit part comprises also the second term in Equation 2.100 which corresponds to an infinite frequency shift. It will annihilate another unphysical term appearing due to the influence of the bath. In the circuit Hamiltonian \( H_{\text{circ}} \), there has to be a capacitive term to guarantee that the node charge is well defined and could involve some flux potential coming from linear inductances or Josephson junctions,

\[
H_{\text{circ}} = \frac{q^2}{2C} + V(\varphi).
\]

**Bath correlation function**

The thermal state of the bath at temperature \( T (\beta = \frac{1}{k_B T}) \) reads

\[
\rho_B = \frac{\exp(-\beta H_B)}{\text{Tr} \exp(-\beta H_B)},
\]

which is diagonal in the number basis of all the bosonic bath modes. The bath operator \( B \) does not commute with the bath Hamiltonian and thus it has a non-trivial evolution in the interaction picture,

\[
B(t) = \exp(iH_B t) B \exp(-iH_B t).
\]
Since the Hamiltonian is time independent, the correlation of the bath operators depends only on the time difference and written in terms of commutator and anti commutator it has the form

\[ \langle B(0)B(\tau) \rangle = \frac{1}{2} (\langle\{B(0),B(\tau)\}\rangle - i \langle [B(0),B(\tau)] \rangle). \]  

(2.106)

Loosey speaking, the bath operator controls the excitation flow between system and bath. Hence, the correlation function determines, if an excitation once having entered the bath can influence the system at another time. For a finite system the correlation function has a recurrence time scale given by the least common multiple of all the bath periods. In turning to an infinite bath, this recurrence times becomes infinite and the correlation functions tend to have an exponential decay. Thus, the time scale of this decay limits the interval during which past interactions still matter and is the basis for the discussion of the Markov approximation.

For a thermal state, the correlation is evaluated, using the two point correlation functions of the mode operators from Equation 2.22 and yields the noise kernel,

\[ D_1(\tau) := \langle\{B(0),B(\tau)\}\rangle = 2\hbar \int d\omega J(\omega) \coth \left( \frac{\beta \hbar \omega}{2} \right) \cos(\omega\tau), \]  

(2.107)

and the dissipation kernel,

\[ D(\tau) := \langle i [B(0),B(\tau)] \rangle = -\int d\omega J(\omega) \sin(\omega\tau), \]  

(2.108)

with the spectral density,

\[ J(\omega) = \sum_k \frac{1}{2L_k^2 C_k \omega_k} \delta(\omega - \omega_k). \]  

(2.109)

Noise and dissipation kernel contain all relevant information about the bath for the system evolution. They only depend on the temperature of the bath and its spectral density.

So far \( J(\omega) \) is not a density, but a collection of delta functions. Only in the limit of the infinite bath, it can be regarded as a continuous function. The spectral density of an ensemble of LC oscillators that mimic a linear resistance is Ohmic, i.e. it grows linearly with \( \omega \) and includes a high frequency cut-off \( \Omega \),

\[ J_{\text{ohm}}(\omega) = \frac{2C\gamma}{\pi} \omega \frac{\Omega^2}{\Omega^2 + \omega^2} \]  

(2.110)

where \( \gamma \) turns out to be the damping frequency of the circuit.

---

8The anti commutator is defined as \( \{A, B\} = AB - BA \).
In this case, the dissipation and noise kernel show exponential decays whose slowest decay rates are the cut-off frequency $\Omega$ and the thermal frequency $\lambda = \frac{k_B T}{\hbar}$ [14, p.176]. The Markov approximation is thus valid in the limit,

$$\gamma \ll \min(\lambda, \Omega).$$  \hspace{1cm} (2.111)

**The Caldeira-Leggett master equation**

Hence, the evolution of the circuit density matrix $\rho(t)$ in the Schrödinger picture can be investigated using the memoryless Born-Markov master equation from Equation 2.99

$$\frac{d}{dt} \rho(t) = -\frac{i}{\hbar} \left[ H_{\text{circ}} + \sum_k \frac{\phi_k^2}{2L_k}, \rho(t) \right] - \frac{1}{\hbar^2} \int_0^\infty d\tau \left\{ \text{Tr}_{\text{I}} \left[ H_{\text{I}}, [H_{\text{I}}(-\tau), \rho(t) \otimes \rho_B] \right] \right\}. \hspace{1cm} (2.112)$$

The second term might be rewritten in terms of the noise and dissipation kernel [14, p.175],

$$\frac{1}{2\hbar^2} \int_0^\infty d\tau (iD(\tau) [\phi, \{\phi(-\tau), \rho(t)\}] - D_1(\tau) [\phi, [\phi(-\tau), \rho(t)]]). \hspace{1cm} (2.113)$$

If the system evolves slowly, i.e. for a typical system frequency $\omega_0$ it holds,

$$\omega_0 \ll \min(\lambda, \Omega), \hspace{1cm} (2.114)$$

in the time range of the kernels $t < \frac{1}{\min(\lambda, \Omega)} \ll \frac{1}{\omega_0}$, the evolution can approximately be separated into its flux and charge contribution,

$$\exp(-iH_{\text{circ}}t) \approx \exp \left( -i \frac{q^2}{2C} t \right) \exp \left( -iV(\phi)t \right). \hspace{1cm} (2.115)$$

Therefore, in this time range the flux evolves freely, $\phi(-\tau) \approx \phi - \frac{q}{2C} \tau$. This slow system limit allows to evaluate the integrals, showing that the counter Hamiltonian is cancelled and finally yielding the Caldeira-Leggett master equation (for details see [14, pp. 177-178]),

$$\frac{d}{dt} \rho(t) = -\frac{i}{\hbar} [H_{\text{circ}}, \rho(t)] - \frac{i\gamma}{\hbar} [\phi, \{q, \rho(t)\}] - \frac{2Ck_B T \gamma}{\hbar^2} [\phi, [\phi, \rho(t)]] \hspace{1cm} (2.116)$$

It can be shown that this master equation is not exactly of Lindblad form [14, p.178]. However, the violation can be repaired by a term that vanishes in the high temperature limit.

[72x724]
Figure 2.6: The spectral density contains the structure of the bath of harmonic oscillators. The Ohmic spectral density $J_{\text{ohm}} = \frac{2C\gamma}{\pi^2} \frac{\Omega^2}{\Omega^2 + \omega^2}$ generates a constant damping. For different damping rates, e.g. different resistors in a circuit, the whole spectral density has to be altered.

Decoherence and dissipation in an LC circuit according to the Caldeira-Leggett master equation

The Caldeira-Leggett master equation for the case of an LC circuit, $H_{\text{circ}} = \varphi^2 / 2L + q^2 / 2C$ with $\omega_0 = \frac{1}{\sqrt{LC}}$, and an initial Gaussian state (see Appendix B) gives the following expressions for the cumulants of $\varphi$ and $q$,

\[
\frac{d}{dt} \langle \varphi \rangle = \frac{\langle q \rangle}{C}, \quad (2.117)
\]

\[
\frac{d}{dt} \langle q \rangle = -\frac{\langle \varphi \rangle}{L} - 2\gamma \langle q \rangle, \quad (2.118)
\]

\[
\frac{d}{dt} V_{\varphi} = \frac{2}{C} C_{\varphi,q}, \quad (2.119)
\]

\[
\frac{d}{dt} V_q = -\frac{2}{L} C_{\varphi,q} - 4\gamma (V_q - Ck_B T), \quad (2.120)
\]

\[
\frac{d}{dt} C_{\varphi,q} = \frac{1}{C} V_q - \frac{1}{L} V_{\varphi} - 2\gamma C_{\varphi,q}. \quad (2.121)
\]

The damping due to the resistance appears in form of a friction term in the average charge evolution. The comparison with the classical circuit equation,

\[
\ddot{q}_c = -\omega_0^2 q_c - \frac{1}{RC} \dot{q}, \quad (2.122)
\]

phenomenologically identifies $2\gamma = \frac{1}{RC}$. The discrete version of the spectral density $J(\omega)$ shows that the rate $\gamma$ is a result of the coupling strength between system and bath. Hence, different resistance values can be engineered by adjusting the coupling constants. This is illustrated in Figure 2.6.

The evolution of the variances leads to thermalisation of the state with the stationary values $V_q = Ck_B T, V_{\varphi} = Lk_B$ and $C_{\varphi,q} = 0$ which correspond to the thermal energy $\frac{k_B T}{2}$. 
Figure 2.7: Evolution of a coherent state according to the evolution of the Caldeira-Leggett master equation. Parameters of the evolution are the damping rate $\gamma = 0.3\omega_0$ and the thermal frequency $\lambda = \frac{k_B T}{\hbar} = 20\omega_0$.

in the electric and magnetic degree of freedom of the circuit. Thermalisation produces a very mixed state and the stationary value of the purity is, see Appendix B,

$$\Pi = \frac{k_B T}{2\hbar \omega} \ll 1.$$  \hfill (2.123)

The low purity comes from the decoherence due to the coupling to the bath. In principle, an observer could use the environment as a probe to learn about the flux in the system by a projective measurement on the bath. Successively performing such measurements, the circuit state would be pushed into a quasi flux eigenstate with a highly squeezed flux variance and a flux value drawn from the probability distribution of the originally prepared state. The Caldeira-Leggett master equation captures the evolution for the density operator of an observer who ignores the state of the environment. This observer measures the probabilistic ensemble of the possible flux eigenstates and thus the flux coherence is suppressed.
For an illustrating evolution, see Figure 2.7. Since the scaling of the figure is used again later, it is reviewed here. For convenience, flux and charge are depicted unitless,

\[
\varphi \rightarrow \hat{\varphi} ; \varphi_0 = \sqrt{\frac{\hbar}{\omega_0 C}}, \\
q \rightarrow \hat{q} ; q_0 = \sqrt{\hbar \omega_0 C}.
\]

Because \( \varphi_0^2 = 2 \langle 0 | \varphi^2 | 0 \rangle \) is twice the vacuum variance, a coherent state, which corresponds to the shifted vacuum, has variances \( V_{\varphi} = \frac{1}{2}, V_q = \frac{1}{2}, C_{\varphi,q} = 0 \). Finally, time is expressed in multiples of the inverse circuit frequency \( t \rightarrow \omega_0 t \). Correspondingly, all frequencies are expressed as multiples of \( \omega_0 \).

### 2.2.2 Continuous measurement

The basic postulates of quantum measurement theory do not involve time. A projective, von Neumann measurement describes an instantaneous reduction of the quantum state to the post-measurement state. Physically, a measurement process happens on a certain time-scale and a finer time-resolution should enable the observation the gradual collapse of the state [15, 37]. Correspondingly, observers can only extract information via measurements at a finite rate.

The continuous flow of information received by the observer allows quantum feedback control: performing an action on the quantum system conditioned upon the past series of measurement records. Usually, quantum feedback control is designed to protect a quantum state or to drive it on a given trajectory [28, ch. 5]. In designing a feedback process, it is crucial to take the perturbation of the state by the measurement into account.

The following section presents the stochastic master equation that captures the effects of a continuous monitoring of a quantum state. First, the form of the measurement output and the structure of the master equation is explained. Second, a brief derivation of this structure based on the representation of quantum dynamics by positive maps is given. It shows how stochastic and deterministic parts of a quantum evolution are necessarily linked. The section concludes with the analysis of a continuous charge measurement of an LC resonator.

#### 2.2.2.1 The stochastic master equation

If an observer performs a measurement on a quantum system which until this point has been evolving unobserved, the density operator is only restricted by the unity of the
trace and positivity. Until the first measurement, the quantum state has evolved uncon-
ditionally. Once the measurement has taken place, the state has to be compatible with
the output of the measurement. Thus, the evolution of a system which is constantly cou-
pied to a measurement apparatus is conditioned on the corresponding record of outputs.
It undergoes a conditioned evolution. In the following, the description of conditioned
evolutions developed by Wiseman and Milburn is reviewed [12, ch. 4].

The main characteristics of a conditioned evolution are already contained in the simplest
case of a projective measurement. The measurement output is probabilistic and the
state of the system will be perturbed by the measurement. Both are intrinsic quantum
effects, the first originating in the superposition principle and the second preventing
the violation of the Heisenberg uncertainty principle. A projective measurement obtains
complete information about the post-measurement state and in return leads to a collapse
of the pre-measurement density operator.

Quantum measurement works in the frame of the information-gain versus perturbation
pay-off. Let us consider an observer who learns about the average of a system operator
$\langle c + c^\dagger \rangle$ via a continuous update,

$$dm(t) = \frac{\langle c + c^\dagger \rangle}{2} dt + \frac{1}{\sqrt{8k\eta}} dW$$  \hspace{1cm} (2.124)

with the Wiener increment $dW$ i.e. the measurement output signal has the form,

$$M(t) = \frac{\langle c + c^\dagger \rangle}{2} + \frac{1}{\sqrt{8k\eta}} \zeta(t),$$  \hspace{1cm} (2.125)

where $\zeta(t)$ is white noise (see Appendix A), $\eta$ denotes the efficiency of the measurement
and $k$ is the measurement strength. The efficiency is a unitless parameter ranging be-
tween 0 and 1 and corresponds to a detector efficiency. A low efficiency detector uses
only a part of the available information and thus for $\eta < 1$, the noise in the measurement
is bigger. In the following, only efficient measurement will be considered. The measure-
ment strength $k$ determines the rate at which the noisy measurement output reveals a
reliable information. A single time measurement output does not allow to reliably learn
the average value $\langle c + c^\dagger \rangle$, because the signal-to-noise ratio vanishes due to $dW = \sqrt{dt}$,

$$\frac{\text{Signal}}{\text{Noise}} \propto \frac{dt}{dW} = \sqrt{dt}.$$  \hspace{1cm} (2.126)

A continuous measurement output with this feature provides a physical case of a measure-
ment in which 'the information obtained goes to zero as the duration of the measurement
goes to zero' [15, p. 4]. Valuable information is contained in the time average of the
above equation during an interval $T$,

$$
\frac{1}{T} \int_{0}^{T} dt' M(t') = \frac{\langle c + c^\dagger \rangle}{2} + \frac{W(T)}{\sqrt{8kT}}.
$$

The diffusive growth of the noise term $\propto \sqrt{T}$ leads to an increasing signal-to-noise ratio. Denoting by $V_0$ the vacuum variance of the measured observable $c + c^\dagger$, a characteristic frequency of the measurement is $\tau = kV_0$. For $T = \frac{1}{\tau}$, the averaging has reduced the noise to its quantum fluctuation limit. Hence, the frequency $\tau$ gives the rate at which the state is quasi projected and will be called the projection frequency.

The evolution of the conditioned state $\rho_M(t)$ corresponding to one realisation of the measurement output is called a quantum trajectory and obeys the stochastic differential equation (SDE),

$$
d\rho_M(t) = -\frac{i}{\hbar} [H, \rho_M(t)] dt + 2kD[c]\rho_M(t)dt + \sqrt{2kH[c]\rho_M(t)}dW.
$$

The first term is the Hamiltonian evolution and the two contributions due to the measurement appear in the form of the Lindblad super operator

$$
D[c]\rho = c\rho c^\dagger - \frac{1}{2} \left\{ c^\dagger c, \rho \right\}
$$

and the non-linear measurement super operator

$$
H[c]\rho = c\rho + \rho c^\dagger - \frac{\langle c + c^\dagger \rangle}{\text{Tr}[\langle c + c^\dagger \rangle \rho]} \rho.
$$

The Lindblad super operator enters into the deterministic part of the evolution. Its role as preserving the uncertainty principle becomes clear when considering an observer who does not have access to the measurement results and thus, observes the unconditioned state which is the average of the ensemble of conditioned states,

$$
\rho(t) = \langle \langle \rho_M(t) \rangle \rangle
$$

Due to the Ito property $\langle \langle \rho_M dW \rangle \rangle = \langle \langle \rho_M \rangle \rangle \langle \langle dW \rangle \rangle = 0$, the unconditioned state evolves according to

$$
\dot{\rho}(t) = -\frac{i}{\hbar} [H, \rho(t)] + 2kD[c]\rho(t)dt.
$$

Thus, the Lindblad super operator $D[c]$ also enters into the unconditioned evolution. In the case of a particle whose position $X$ is continuously monitored, the Lindblad contribution leads to a decoherence which in the position representation suppresses the
non-diagonal entries,
\[
\frac{d}{dt} \rho_{x,x'}(t) = -\frac{k}{2} (x - x')^2 \rho_{x,x'}(t)
\]
Correspondingly, in the momentum representation, it leads to a diffusion
\[
\frac{d}{dt} \rho_{p,p'}(t) = \frac{k}{2} \left( \partial_p^2 - 2 \partial_p \partial_{p'} + \partial_{p'}^2 \right) \rho_{p,p'}(t).
\]
This diffusion is necessary to preserve the uncertainty principle. Otherwise, two observers could do independent measurements of the two conjugate variables \(X\) and \(P\) and by later on exchanging their results beat the Heisenberg limit.

The second stochastic term corresponds to a non-linear drift term reflecting the information gain due to the measurement [38, p. 5] and thus, only appears in the evolution of the conditioned state \(\rho_M(t)\).

### 2.2.2.2 Derivation of the stochastic master equation

The continuous master equation can be derived in multiple ways. There are microscopic derivations based upon specific physical systems including a model of the detector, like an atom in a leaky cavity whose lost photons are registered by photodetectors [15, pp. 14-21]. An advantage of a microscopic derivation consists in clarifying how the measurement strength \(k\) and the finally measured observable can be engineered. They are an effect of the form of the interaction Hamiltonian between system and the measurement probe and the basis in which the final von Neumann measurement on the probe is performed [12, ch. 4]. In these microscopic models, the origin of the white noise is a Markovian, time coarse-graining approximation.

In this section, an abstract approach to the stochastic master equation is presented that allows a general justification of its form. It is based on taking the complete positivity of the evolution as a starting point and requiring a Wiener increment contribution to the dynamics. Such a derivation has for example been developed in [15, pp. 10-12]. This section contains a short summary in order to present the inevitable link between the stochastic and the deterministic part of the evolution. Here, the conditioned state is \(\rho\) and the unconditioned state is \(\bar{\rho}\).

Let us assume an infinitesimal state update for the conditioned state \(\rho\) of the form required by complete positivity,
\[
d\rho = A\rho A^\dagger - \rho,
\]
(2.133)
where $A$ contains a deterministic and a stochastic part,

$$A = 1 - i \frac{H}{\hbar} dt + b dt + c dW.$$

The deterministic part is proportional to $dt$ and consists of the anti-Hermitian part $iH$ and the so far unspecified Hermitian part $b$. The stochastic part is proportional to the Wiener increment $dW$.

Due to the structure of the completely positive map and the formal order $\sigma(\sqrt{dt})$ of the Wiener increment, the conditioned state update up to order $dt$ reads,

$$d \rho = \left[ -i \frac{H}{\hbar} [H, \rho] + \{b, \rho\} + c \rho c^\dagger \right] dt + \left( c \rho + \rho c^\dagger \right) dW.$$

(2.135)

In the unconditioned evolution only the deterministic part survives,

$$d \bar{\rho} = \left[ -i \frac{H}{\hbar} [H, \bar{\rho}] + \{b, \bar{\rho}\} + c \bar{\rho} c^\dagger \right] dt.$$

(2.136)

The trace preservation condition, $\text{Tr} d \bar{\rho} = 0$ establishes the previously mentioned link between the stochastic and the deterministic part of the evolution,

$$b = -\frac{c^\dagger c}{2}.$$

(2.137)

This link allows to rewrite the unconditioned evolution in terms of the Lindblad super operator,

$$d \bar{\rho} = \frac{i}{\hbar} [H, \bar{\rho}] dt + D[c] \bar{\rho} dt.$$

(2.138)

As commented earlier, this is the general form for the Markovian evolution of an open quantum system and the direct derivation of its structure was the original motivation for the above argumentation [39]. This also adds a valuable viewpoint of quantum dynamics governed by a Lindblad super operator as the unconditioned evolution of some underlying monitoring process, which is for example used in the numerical simulation of open quantum systems [12, p. 153].

Returning to the conditioned evolution, the preservation of the trace prescribes

$$\text{Tr} \left( (c + c^\dagger) \rho \right) = 0.$$

(2.139)

Instead of understanding this as a condition on the operator $c$, trace preservation is restored by subtracting a term $\langle c + c^\dagger \rangle \rho dW$ from the evolution. This renormalisation adds the non-linear part to the evolution and thus, after introducing the measurement strength by rescaling $c \rightarrow \sqrt{2k}c$, the stochastic master equation from Equation 2.128 is recovered.
2.2.2.3 Monitoring charge in an LC circuit

Let us consider an LC circuit coupled to an ideal voltmeter which measures the mean of the voltage between the top and the ground node. Apart from the physical details of this voltmeter, the evolution of a quantum state in the LC circuit is perturbed by the presence of the measurement device. Since the voltage is proportional to the charge on the capacitor, the monitoring gradually reveals information about the charge mean. If the voltmeter output is of the form in Equation 2.125,

\[
M_V(t) = \langle V \rangle(t) + \frac{1}{\sqrt{8kV}}\zeta(t),
\]

which rescaled by the capacitance \(C\) reads

\[
M_q(t) = \langle q \rangle(t) + \frac{1}{\sqrt{8k}}\zeta(t),
\]

the quantum state evolves corresponding to the SME in Equation 2.128 with the operator \(c = q\) and measurement strength \(k = \frac{k_V}{C^2}\). The assumption of Wiener noise in the measurement outcome is very general due to the central limit theorem. Any measurement noise with bounded variance becomes Wiener noise in a time coarse-graining [15].

As a preparation of Chapter 4 the evolution of the cumulants of a Gaussian state in the monitored LC circuit is studied, a calculation done for example in [15, p. 11-12]. With the same rescaling of charge, flux and time as in the discussion of the Caldeira-Leggett master equation in Section 2.2.1.2 and the definition of the charge projection frequency, \(\tau = kq^2\), the SME for the conditioned state \(\rho_M\) reads,

\[
d\rho_M = -\frac{i}{2} [q^2 + \varphi^2, \rho_M] \, dt - \tau [q, [q, \rho_M]] \, dt + \sqrt{2\tau}(\{q, \rho_M\} - 2 \langle q \rangle \rho_M) dW. \tag{2.142}
\]

For the evolution of the expectation value of an operator \(A\), this implies

\[
d \langle A \rangle = \operatorname{Tr} (Ad\rho_M) = -\frac{i}{2} \langle [A, q^2 + \varphi^2] \rangle \, dt -\tau \langle [q, [q, A]] \rangle \, dt + \sqrt{2\tau} (\{q, A\} - 2 \langle q \rangle \langle A \rangle) \, dW. \tag{2.143}
\]

It is straightforward to get the evolution of the average flux and charge,

\[
d \langle \varphi \rangle = \langle q \rangle \, dt + \sqrt{8\tau} C_{\varphi,q} dW; \tag{2.144}
\]
\[
d \langle q \rangle = -\langle \varphi \rangle \, dt + \sqrt{8\tau} V_q dW. \tag{2.145}
\]

However, the calculation of the variances requires some care, because the conditioned
evolution is a stochastic differential equation in the Ito form. Thus, the usual rules of differentiation do not apply, see Appendix A. For example, in order to calculate the charge variance increment,

\[ dV_q = d\langle q^2 \rangle - 2 \langle q \rangle d\langle q \rangle - (d\langle q \rangle)^2, \]  

(2.146)

the last second order term has to be included due to the \( \sigma(\sqrt{dt}) \) order of the Wiener increment. Evaluation of this expression up to order \( dt \) yields,

\[ dV_q = (-2C_{\varphi,q} - 8\tau V_q^2) dt + \sqrt{8\tau} \left( \langle q^3 \rangle - \langle q \rangle^3 - 3 \langle q \rangle V_q \right) dW. \]  

(2.147)

If the initial state was not Gaussian, the last stochastic part would be non-zero and also the evolution of higher moments would have to be included. For Gaussian states all moments are functions of the variances and are related in such a way that the stochastic part cancels, see Appendix B. In the same manner, the flux variance and the covariance are obtained. Hence, the evolution of the second cumulants is deterministic and reads

\[
\begin{align*}
\frac{d}{dt} V_\varphi(t) &= 2C_{\varphi,q} + 2\tau(1 - 4C_{\varphi,q}^2), \\
\frac{d}{dt} V_q(t) &= -2C_{\varphi,q} - 8\tau V_q^2, \\
\frac{d}{dt} C_{\varphi,q}(t) &= (V_q - V_\varphi) - 8\tau C_{\varphi,q} V_q.
\end{align*}
\]

(2.148) \quad (2.149) \quad (2.150)

These represent a Ricatti equation for the vector \((V_\varphi, V_q, C_{\varphi,q})\) and are analytical solvable [40, p. 2704]. The stationary values read

\[
\begin{align*}
C_{\varphi,q}^{st} &= -\frac{1}{8\tau} \left( \sqrt{1 + (4\tau)^2} - 1 \right), \\
V_q^{st} &= \frac{1}{2} \sqrt{\frac{1}{\tau} |C_{\varphi,q}^{st}|}, \\
V_\varphi^{st} &= \sqrt{1 + (4\tau)^2 V_q}.
\end{align*}
\]

(2.151) \quad (2.152) \quad (2.153)

The stationary values reflect, which variable has been measured. With increasing measurement strength \( k \), i.e. projection frequency \( \tau \), the final state is more and more localised in charge space, whereas the flux variance increases due to the uncertainty principle. The final conditioned state is a pure and squeezed. The purification can be seen by calculating the evolution of the determinant of the covariance matrix,

\[ \frac{d}{dt} \det \sigma = -8\tau V_q \left( \det \sigma - \frac{1}{4} \right), \]  

(2.154)

whose value \( \det \sigma \geq \frac{1}{4} \) determines the purity of a Gaussian state, \( \text{Tr} \rho_M^2 = \frac{1}{2\sqrt{\det \sigma}} \).

Figure 2.8 shows a quantum trajectory in the monitored LC circuit. Note the white
Figure 2.8: Conditioned evolution of a charge-monitored LC circuit with $\tau = 0.8$. The first moments fluctuate around the free oscillation due to the stochastic back action of the measurement (top left). Whereas charge variance becomes reduced, the flux variance correspondingly rises to preserve the uncertainty principle (top right). The measurement output is corrupted by white noise that only upon a time window average reveals the charge evolution (bottom left). The energy also fluctuates due to the random kicks of the first moments by the measurement and the initially pure state stays pure (bottom right). Initial conditions are $\varphi(0) = 5, q(0) = 0, V_{\varphi} = 0.5, V_q = 0.5, C_{\varphi,q} = 0, \mu(0) = 0$ and the time evolution was simulated with an explicit Euler Algorithm $dt = 10^{-3}$.

noise in the measurement output, revealing the vanishing information gain in a single time step. Only after averaging over a sufficient time interval, a reliable information about the charge evolution can be recovered.

2.2.3 Measurement-based feedback control of quantum systems

Nowadays, experimentalists have powerful tools to act in a controlled manner on individual quantum systems. The two main tools at hand are measurements and coherent control pulses which shape the Hamiltonian governing the system. Usually, this two tools are used separately. A typical experiment consists of an initialisation step preparing a known quantum state, a series of control pulses and a final measurement. Such a control is blind to the quantum state. It cannot adapt to perturbations of the state due to uncontrolled environmental couplings.
A more general control scheme is measurement-based quantum feedback control, combining measurement and control [12, 28]. Quantum systems can be continuously monitored and the observer is able to use the obtained information either in the subsequent control pulses or in an adaption of the measurement process. The main difference to feedback control in classical systems is the measurement process. In principle, a classical system can be monitored without perturbing it, while a quantum state unavoidably is altered by an observation. As a consequence, it is essential to incorporate this back action into the design of any quantum feedback mechanism and even enriches the variety of control mechanisms by exploiting it [41].

The setup for measurement-based quantum feedback control is a quantum system coupled to a measurement apparatus and a control device. The measurement apparatus has a classical communication line to the control device which consists of a set of actuators that control the system Hamiltonian and/or the settings of the measurement. The control device includes a memory to register the past measurement records and a processing unit which takes the content of this memory as an input to calculate, which control has to be performed. Such a setup is schematically shown in Figure 2.9.

Starting point for considering the feedback is the conditioned evolution of the monitored system in Equation 2.128,

\[
d\rho_M(t) = -\frac{i}{\hbar}[H, \rho_M(t)]dt + 2kD[c]\rho_M(t)dt + \sqrt{2k}(c)\rho_M(t)dW,
\]

(2.155)
and the corresponding measurement output,

\[ M(t) = \frac{\langle c + c^\dagger \rangle}{2} + \frac{1}{\sqrt{8k\eta}} \zeta(t). \]

Furthermore, the feedback introduces an additional update of the density operator which in general has the form,

\[ [\dot{\rho}_M(t)]_{fb} = \mathcal{F}[t, M(s)_{t_0 \leq s < t}]\rho_M(t). \]  \hspace{1cm} (2.156)

The super operator \( \mathcal{F} \) is a function of time and a functional of the measurement record up to the present time \( t \). The latter characteristic allows a control depending on the history of measurement records. Due to the Wiener noise in the measurement record, this implicit equation has to carefully transferred to the explicit state increment \( d\rho_M(t)_{fb} \). The full evolution is obtained by adding the two contributions, Equation 2.155 and 2.156. This simple adding works, because the noise due to the measurement at time \( t \) and the noise in the feedback update, which only includes measurement records up to \( t \), are independent. Otherwise, in the case of Markovian feedback, the special case where the above feedback term only involves the measurement result at \( t \), time ordering of the two processes, measurement and feedback, is essential [12, p. 239].

In a last step, the unconditioned evolution is obtained by ensemble average of the resulting master equation involving the explicit feedback term. Wiseman and Milburn comment that this ensemble average might not yield a closed equation for the unconditioned state [12, p. 239]. They say that if the feedback makes use of past measurement results, the so generated non-linear and non-Markovian conditioned evolutions prohibit a master equation for the unconditioned state.
2.3 Memory elements in circuits

If an electrical engineer were provided with a set of linear resistors, capacitors and inductors to build an electric circuit version of the brain, he would immediately protest. No matter how much effort he would put in the circuit design, at the end, the input-output relation he could establish would always be characterised by a simple impedance. The being equipped with such a linear brain would not be able to make a simple Yes-No decision which would require a non-linear processing step. In a second try, he is offered non-linear resistances, capacitances and inductances and again he complains. Though a non-linear brain could solve decision problems, it would be cursed to decide in the same way forever. A being, supplied with a brain whose circuit design had been optimised for one environment, would helplessly fail in a surrounding with different requirements. It would lack the capability of learning whose first prerequisite is memory which consists in a storage for previous sensations and whose state alters the input-output relationship.

The biologists Hodgkin and Huxley discovered the limitations of the resistor, capacitor, inductor set when they tried to design a circuit mimicking the behaviour of the membrane current in neurons [1]. In order to generate the observed current-voltage behaviour, they modelled the ion channels which control the charge flow across the membrane by ‘dynamical resistances’. These resistors have an internal degree of freedom which gives them a memory capability in the sense of a response dependent of the voltage history.

In 1971, Chua discovered the memristor, a passive circuit element that behaved like a ‘dynamical resistance’ [2]. He observed that a circuit element with a flux-charge relationship constitutes a non-linear resistor with a resistance value depending on the amount of charge which has crossed it,

$$\varphi(t) = M(q(t)) \quad \Rightarrow \quad V(t) = \frac{\partial M(q)}{\partial q} I(t).$$

In this way the memristor combines a storage, which registers input (currents), and a processing device turning the input into an output signal (current to voltage).

Providing the electrical engineer with such a circuit element would enable him to close the gap of memory and adaptation and go one step further in establishing a circuit version of the brain. In fact, circuits supplied with memory circuit elements characterised by a non-linear history-dependent response have been demonstrated experimentally [3] and besides offering a natural implementation of learning circuits [42], they inspired a new computing architecture, the universal memcomputing machine [5].

The key feature of memory circuit elements, the combination of memory and processing, could also increase the tools available in quantum electrical circuits. Naturally, this
poses the exciting question whether such tools could enable a symbiosis of the two novel
information processing paradigms, quantum computing and memcomputing.

In this section, first, memory circuit elements are defined via their constitutive rela-
tionships. Subsequently, focusing on memristive elements the physical limitations connected
to this formulation are reviewed. Second, a short overview of the applications of memory
circuit elements in classical computing is given. Finally, a Lagrange approach to memory
elements in circuits is used to discuss a possible quantisation scheme, concluding that
such a frame leaves behind the gap of the quantum version of the memristor [9].

2.3.1 Memristors, memcapacitors and meminductors

Memory elements in circuits are characterised by relating two circuit variables from the
set of charge, flux, current and voltage via a response function $g$ that also depends on
a further set of variables $x$ [9, p. 1]. These are called the state variables and represent
internal degrees of freedom of the element. For a control $u(t)$, e.g. a voltage signal, the
response $y(t)$, e.g. the current, is given by the two equations,

$$y(t) = g(x(t), u(t), t)u(t),$$  \hspace{1cm} (2.157)
$$\dot{x}(t) = f(x(t), u(t), t).$$  \hspace{1cm} (2.158)

Besides the dependence of $g$ on the state variable $x$, the essential structure empowering
memory is a non-trivial dependence of $f$ on the control $u(t)$. In this case, the solution
of the second equation contains a time integral over the past of the control signal. Since
this solution enters into the response function $g$, the output $y(t)$ depends not only on
the present value of the control, but also on the previous ones. Hence, if a memory
circuit element is tested by applying a periodic control signal, a hysteresis in the $y - u$
plane is observed, the footprint of memory. While an $I$-$V$-hysteresis gives the class of
memristors, a $q$-$V$-hysteresis characterises a memcapacitor and a history-dependent $\varphi$-$I$
relationship defines a meminductor [43]. The corresponding circuit elements are shown
in Figure 2.10.

In the following, the focus is on the class of memristors. Since the elements with an
$I$-$V$ relationship in form of Equation 2.157 and Equation 2.158 go beyond Chua’s flux-
charge memristor, in the literature often the term memristive elements is used [9]. These
elements have features like the explicit time dependence of the resistance value which are
not essential for the memory property. Hence, for the purpose of this thesis a simplified
definition of the memristor adapted from Lehtonen [4] is used.
The classical voltage-controlled memristor is defined as a circuit element with an $I$-$V$ response of the form,

$$I(t) = G(\mu(t))V(t),$$

$$\dot{\mu}(t) = f(\mu, V(t)),$$

with the following conditions for the two continuous real valued functions $f, G$:

\begin{itemize}
\item[(i)] for all values of $\mu : G(\mu) \geq 0$
\item[(ii)] for a fix $\mu$ the function $f(\mu, v)$ is monotone and $f(\mu, 0) = 0$.
\end{itemize}

Property (i) ensures that $G(\mu)$ can be understood as a conductance and guarantees passivity of the element. Property (ii) restricts the state variable dynamics in order to provide a non-volatile and efficient memory. Non-volatility ensures that the state variable does not change, if the applied voltage is zero. Efficiency of the memory should imply that all significant voltage inputs are registered and is guaranteed by the monotonicity of $f$. Hence, the definition by Lehtonen has been changed in two ways. On the one hand, the conductance $G$ is only a function of $\mu$ and not of $V$, which restricts the non-linearity of the voltage-current relationship to the indirect one due to the state variable. On the other hand, in the present definition, $G(\mu)$ is not restricted to be an increasing function in order to include the memristive term appearing in the RSJ model of the Josephson junction. Relaxing this restriction means that multiple crossing hysteresis loops are allowed.

The memristor indeed extends the set of passive circuit elements, because it is a fundamental circuit element which cannot be replaced by any network of capacitors, inductors and resistors [4, Theorem 7].
Figure 2.11: The left figure shows the dependence of the conductance on the state variable for the memristor in the RSJ model. The black point corresponds to the ideal initial condition for the state variable $\delta(t)$ to observe a clear hysteresis loop. The ratio $\frac{2e}{\hbar}V_0$ determines the change of the memory variable, which is represented by the arrows for three different driving signal amplitudes with the corresponding hysteresis curves in the right figure. Note that the endpoints of the arrows mark the state variable value, when the $I$-$V$ curve crosses the origin. Due to the symmetry of the conductance around $\delta = 0$, an initial condition at the minimum or maximum of the conductance function will not show hysteresis (red).

Example: As an illustration, let us consider the memristor in the RSJ model of a Josephson junction, see Section 2.1.1.2,

$$I(t) = G[1 + \epsilon \cos \delta(t)] V(t), \quad (2.161)$$
$$\dot{\delta}(t) = \frac{2e}{\hbar} V(t). \quad (2.162)$$

Hence, the superconducting phase difference across the junction $\delta(t)$ works as the state variable. A periodic voltage signal $V(t) = V_0 \sin(\omega t)$ drives a current that follows a hysteresis. The emphasis of the hysteresis relates to the ability of the state variable to register the evolution of the voltage signal. It only registers the signal, i.e. changes significantly, if the generalised flux $\frac{V_0}{\omega}$ generated during half a period by the input signal $V_0$ is comparable to the flux quantum $\Phi_0 = \frac{h}{2e}$. If it is a multiple of the flux quantum, the periodicity of the memconductance $G(\delta)$ leads to multiple crossings. The different cases are depicted in Figure 2.11.

Memory elements in circuits seem an exception, but they are actually the rule [44]. The description of circuit elements by response functions has a microscopic origin based on internal degrees of freedom. For a model of a resistor, this might be position $R$ of the atoms with mass $M$ in the material. Applying a voltage across the device leads to a
response that depends on these variables and eventually also their velocities,

\[ I(t) = G \left( R, \frac{dR}{dt} \right) V(t). \] (2.163)

Besides, the external electrical field due to the applied voltage affects the dynamics of the internal degrees of freedom,

\[ \frac{d^2 R}{dt^2} = \frac{1}{M} F \left( R, \frac{dR}{dt}, V(t) \right). \] (2.164)

Taken together, these two equations describe a memristive element. Still, the memory effects in the majority of circuit elements are not observable. Due to high relaxations rates, memory only plays a role for inaccessible high-frequency ranges. The reason to consider memory elements as a different class of circuit elements is the stable and controllable character of their state variables, which encode collective, effective properties of the underlying microscopic degrees of freedom. An example is the thin-film memristor of Hewlett-Packard [3] with a state variable which is the border between a non-doped (highly resistive) and a doped (low resistive) region in a semiconductor. The total resistance is the sum of the low and high resistance value weighted with the size of the corresponding regions. In an electrical field the dopants drift and as a joint effect the border is moved resulting in a change of the total resistance, see Figure 2.12.

From a physical point of view, many memristor models are over-simplified to provide a description in terms of Equation 2.157 and Equation 2.158. On the one hand, this a necessary step to explore the consequences of memory effects in complex circuits involving multiple memristors. On the other hand, these simplified models have raised criticism, because they often ignore physical constraints like symmetries of electrodynamics or the inevitable thermal fluctuations of the input signal [44, p. 4-5].

Especially interesting for the information-encoding nature of the state variable is a criticism passed by the authors of [45]. If the voltage is turned off, the state variable should stay constant to preserve the corruption of the information about the history of the previously applied voltage. Hence, Landauer’s principle demands the state variable value to

\[ d\mu \]

\[ \text{Low resistance} \quad \text{High resistance} \]

**Figure 2.12:** Scheme of the HP memristor showing how the state variable changes due to the ion drift that enlarges or shrinks the contribution of the low resistance part
correspond to an equilibrium state, energetically sufficiently separated from other values to withstand thermal agitations. In return, deleting the information, i.e., changing the state variable needs a minimal amount of energy. However, the heat dissipated in a memristor during a change $d\mu$ of the state variable in a memristor is not limited from below,

$$\Delta E = I(t)V(t)dt.$$  

(2.165)

This violation of the Landauer principle implies that the information encoded in the state variable is very sensitive to noise and may only be trusted on time scales shorter than the typical diffusion time. While for the thin-film memristors these times have been reported to be of the order of multiple days [46, p. 19], a proposal of a memristor using the JJ estimates a corruption of the state variable in the nanosecond scale, which would require extremely fast control to be useful [22].

2.3.2 Information processing with memory elements

Information storage and processing with memristors has two levels. In the first level, one focuses on the manipulation and control of the state variable to encode bitwise or analog information in the resistance value, see Figure 2.13. The second level lifts the applied electrical signals from mere control pulses to information encoding entities and enables simultaneously the storage capability and the information processing power of the memristor. It is this second view that suggests the comparison of memristors with the synaptic connections in the brain which combine the functions of a memory unit, a communication link and a processing device [4, p. 3]. Hence, the memristor is considered as the elementary building block of neuromorphic circuits mimicking features of the information processing in the brain like associative memory, learning and pattern recognition [47]. In a further abstraction, this view has inspired an alternative computing architecture, the universal memcomputing machine (UMM) [5, 48].

In the first spirit, memristors offer new storage technologies like the crossbar architecture, have been considered in hybrid on-chip architectures [46] and the concept of stateful logic operations has been developed to perform boolean operations on bits encoded in memristors [49]. Although offering promising technical advantages like very dense storage, low power consumption and eventually lower costs in production, using memory elements in this way does not present a conceptual change in information storage or processing. It neglects their capability of combining these two processes.

The intriguing way of using memristors takes advantage of this capability. A very illustrative example of exploiting the combination of processing and storage in the same
Figure 2.13: The form of the hysteresis curve prescribes the kind of information a memristor is suited to encode. If two resistance values dominate the state variable can carry digital information in form of one bit, whereas a memristor with a hysteresis curve where the resistance is driven in a continuous manner serves to encode analog information.

The physical device is the memristive processor to solve complex mazes [50]. A brief review of the this proposal also highlights the general properties of the UMM.

The memristive processor is a network of memristors encoding the maze (see Figure 2.14). All the memristors are current-controlled and their $I$-$V$ relation has the form,

\begin{align}
V_{ij}(t) &= [R_{\text{low}}x_{ij} + R_{\text{high}}(1 - x_{ij})] I_{ij}(t), \\
\dot{x}_{ij}(t) &= \alpha I_{ij}(t),
\end{align}

where $i,j$ denote the two grid points which are connected by the memristor with the internal state variable $x_{ij}$. The value of $x_{ij}$ is restricted between 0, 1 (thus the second relation is only valid for $x \neq 0, 1$). The memristors are initially in the high resistance state. If a constant voltage is applied between the entrance and the exit of the maze, current only flows through the memristors on the solution paths of the maze. The internal state variable evolves due to this current and finally the memristors on the solution path are in a low resistive state. Thus, the solution path is obtained by reading the individual resistance values of the memristors.

The four main features of the UMM, namely computation and storage on the same platform, intrinsic parallelism, information overhead and functional polymorphism, are present in the memristive processor. For a detailed description of the memcomputing
architecture and an elaborated discussion of its features, see the original paper by Di Ventra [5].

**Computing and storing on the same platform**

The computation of the solution path (i.e. changing the resistance) and the storage of the solution (i.e. final resistance) both happen in the memristive processor. On the contrary, a network of resistors could solve the same task, but would require local current measurements during the computation of the solution and an external memory to store it.

**Intrinsic parallelism**

The state variables of the memristors are updated simultaneously. Computation and storage happen in a single step.

**Information overhead**

The constituents of the network are \( n \) memristors. Thinking of them as encoding bits for a moment, the network has \( 2^n \) possible states. However, the amount of information accessible in the network is much larger, when taking into account the paths in the network that directly encode sums of bits in the individual memristors.

**Functional polymorphism**

Polymorphism is the ability of a machine to compute different functions with the same machine components. Imagine two voltage pulses encoding two classical bits \( x, y \), which enter a circuitry with memelements. Due to the internal dynamics of the memelements, one can picture that, either by choosing different voltage amplitudes or by a previously-sent bit pair, the circuit can change such that it conditionally computes different boolean functions. In the case of the maze, the potential polymorphism is not directly used, but as the authors claim the memristive processor could yield further applications e.g. in the travelling salesman problem.

These features also mark the main differences of the usual Von Neumann architecture to the memcomputing architecture. Due to the intrinsic parallelism and the information overhead, the UMM is able to efficiently solve problems that are in the NP complexity class. This has been demonstrated for the NP-complete subset-sum problem, see [5, 51]. In the same way the universal quantum Turing machine exploiting the massive parallelism, enabled by the superposition principle, provides examples of more efficient algorithms like the Shor algorithm for prime factorisation [29].

A natural question is whether quantum mechanics also allows for the same powerful combination of simultaneous information processing and storage. The nature of memory
as extracting and feeding back information seems to hint towards a pay-off between coherent superpositions and memory effects. However, moving along this pay-off line, there might be some advantageous terrains and the research field of non-Markovian evolutions is exploring parts of this domain.

2.3.3 Quantisation of memory elements in circuits

A concrete approach to address this question consists in extending the quantisation schemes for electrical circuits to include meminductors, memcapacitors and memristors. Yet, the routine way of identifying a Lagrangian for a closed system, can only work, if circuit and state variable dynamics are second-order differential equations without friction terms. Otherwise, in order to reproduce the equations of motion, dissipation potentials or Lagrange multipliers have to be added that do not suggest a direct quantisation procedure. Obviously, this excludes circuits that involve a memristor.

Di Ventra and Pershin developed a Lagrange approach to memory circuit elements in [9] to explore ’whether true quantum effects can be associated with the memory of these systems and which phenomena could emerge from the quantisation of memory elements’.

Their formalism is based on the Lagrange equations of motion,

\[
\frac{d}{dt} \frac{\delta L}{\delta \dot{y}} - \frac{\delta L}{\delta y} = -\frac{\delta R}{\delta \dot{y}}. 
\]  

(2.168)
Here, \( y \) is either a circuit variable or the state variable of a memory element. The term on the right side is a dissipation force generated by a dissipation potential \( R \). Depending on the memory element, this potential is necessary to reproduce equations 2.157 and 2.158 via the above Lagrange equations.

An example of a dissipationless memory circuit element is a memcapacitor given by two metal plates with mass \( M \) of which one is mobile and coupled to a spring with strength \( k \). The spring elongation \( x \), unitless in multiples of the equilibrium distance \( d_0 \) of the two plates, determines the capacitance \( C(x) = \frac{C_0}{1+x^2} \). Connecting the memcapacitor to a voltage source \( V(t) \) results in a Lagrangian,

\[
\mathcal{L}(q, \dot{q}, x, \dot{x}) = qV(t) - \frac{q^2}{2C(x)} + \frac{1}{2}Md_0^2x^2 - \frac{1}{2}kd_0^2x^2.
\]  

(2.169)

The corresponding Euler-Lagrange equations reproduce the two equations of a memcapacitor and show that the spring elongation is the state variable of the memcapacitor

\[
q(t) = C(x(t))V(t),
\]

(2.170)

\[
\ddot{x}(t) = -\frac{k}{M}x + \frac{q^2}{2Md_0^2C(x)^2} \frac{\partial C}{\partial x}.
\]

(2.171)

Circuits containing memory elements without dissipation are quantised in a straightforward way. An example is an LC resonator in which the capacitor has been replaced by the above memcapacitor. The Lagrangian of this circuit is obtained by adding the flux contribution of the inductor, \( \frac{1}{2}Lq^2 \), and removing the voltage source term in Equation 2.169. The transformation to a Hamiltonian and the expansion, \( \frac{1}{C(x)} \approx \frac{1}{C_0}(1 - x) \), show that the circuit is equivalent to a set of two harmonic oscillators. One harmonic oscillator corresponds to the circuit variable non-linearly coupled to the other one, which stands for the state variable. Thus, in lowest order of \( x \) and a rotating wave approximation, the Hamiltonian expressed in terms of the corresponding ladder operators, \( a \) for the circuit and \( b \) for state variable, reads [9, p. 11]

\[
H = \hbar\omega_{LC_0}a^\dagger a + \hbar\omega_{spring}b^\dagger b + g(a^2b + a^2b^\dagger).
\]

(2.172)

Another class of memory elements where there are tools available for a quantum description are characterised by a dissipation potential that only depends on the state variables, e.g. in the case of the memcapacitor with a spring exposed to a velocity dependent friction force. In the equation of motion, friction terms are generated by adding a Rayleigh

\footnote{Note that the second order equation might be read as a first order equation of the vector \( X(t) = (x(t), \dot{x}(t)) \) and hence reproduces the form of Equation 2.158.}
dissipation potential to the description,

\[
\mathcal{R}_{\text{mem}} = \frac{\kappa x^2}{2}.
\]  

(2.173)

As discussed in the previous section on open quantum systems, this type of dissipation in form of a friction term is generated by coupling the spring to an infinite bath of harmonic oscillators and the dynamics could be discussed using the previously mentioned techniques like a master equation.

However, memristors generate non-linear, history-dependent dissipation in the circuit and the state variable dynamics is a first order differential equation, which makes the Lagrange approach unsuitable for quantisation. Let us illustrate this by looking at the form of the dissipation potentials, which generate the voltage-controlled memristor equations 2.159 and 2.160,

\[
\mathcal{R}_{\text{circ}} + \mathcal{R}_{\text{mem}} = \frac{G(\mu)^{-1}q^2}{2} + S_0 \dot{\mu} \left( \frac{1}{2} - g(V(t), \mu) \right).
\]  

(2.174)

where to have consistent units, the state is assumed to be unitless, an action \( S_0 \) is introduced and it is remarked that the function \( g \) has the units of time. If a voltage source is included, \( \mathcal{L} = qV(t) \), the Lagrange equations yield

\[
\dot{q}(t) = G(\mu)V(t),
\]  

(2.175)

\[
\dot{\mu}(t) = g(V(t), \mu).
\]  

(2.176)

In spite of mathematically reproducing the memristor equations, these coupled dissipation potentials lack a physical interpretation and, in particular, do not offer a direct way to think about a joint quantum system of circuit and state variable, coupled to one or more baths, which are able to produce these complex dissipative dynamics.

To sum up, the Lagrange approach to memory elements enables to study non-dissipative memory elements. Furthermore quantises the basis to treat memory elements with a dissipative memory by going to open quantum systems. Yet, it did not provide a way to study quantised circuits involving memristors. In fact, the authors in [9] remark that they `leave the Hamilton formalism for dissipative memory elements to a further publication’. However, in general the Lagrange approach already constitutes a choice in the quantisation of a circuit with memory elements, because it assigns a quantum nature to the state variable itself. On the one hand, a quantum state variable of the memory element does not seem necessary to generate memory effects in the quantised circuit. On the other hand, especially the state variables of physical memristors are
often collective effects and thus, might still only carry classical information, even if the electrical signals in the circuit are quantised.
Chapter 3

Quantum memristor model

The memristor is a promising element in the toolbox of electrical circuits. Due to its information processing features, it is interesting to study the response of a quantum circuit, when coupled to a memristor. Especially, the history-dependent dissipation will induce memory effects in the circuit, leading to what we call a quantum memristor.

The abstract description of a memristor by a state-dependent response function and the state variable dynamics results in a dead end, when trying to establish a Lagrange approach to quantisation. At the same time, this description offers a view of a memristor as a tuneable resistor controlled by a measurement device. This view suggests an extension of the quantum model for a linear resistance by Devoret to a quantum memristor by adding a feedback scheme. In order to induce the history-dependent dissipation, a more complex environment composed of measurement device, feedback line and bath is considered. On these grounds, a master equation for a quantum memristor is proposed.

First, the memristor equations are reviewed in terms of an information extraction and an actuation step. Hence, its dynamics might be simulated by a feedback scheme using measurement records to control a tuneable resistor. Based on this, a phenomenological master equation for a circuit coupled to a memristor is established, and the underlying assumptions about the time scales of measurement, feedback and bath dynamics are discussed. Finally, a road map to a rigorous derivation of this master equation in the formalism of measurement-based feedback is sketched.

3.1 Measurement and feedback model of a memristor

In a black box approach to a memristor, an external observer tests the two port device by applying a voltage signal and measuring the generated current. Having no means
Figure 3.1: The hysteresis in the response of any voltage controlled memristor can be mimicked by a tuneable resistor controlled by an internal observer with access to an ideal voltage measurement and an internal processing unit.

to probe the memristor’s internal structure, the observer can only access the changes due to the dynamics of the internal state variable via the response curves. However, the same response functions are generated, if the memristor is replaced by a tuneable resistor controlled by an internal observer, who has access to a voltage measurement, see Figure 3.1.

The protocol works as follows: The internal observer first updates his knowledge via the voltage measurement, afterwards calculates the update of the state variable given by Equation 2.160

$$\dot{\mu}(t) = f(\mu, V(t)),$$

and finally, in a feedback step, turns the control knob of the tuneable resistor to the value $R(\mu) = G(\mu)^{-1}$. If the observer does this sufficiently fast and his voltage measurement device is ideal, he is able to fool the external observer and simulate any memristor. Here, sufficiently fast means below the time resolution of the external observer and an ideal voltmeter has an infinite resistance, so that no current flows through it.

The description above holds for classical circuits. If the external observer cools down the system, the quantum realm emerges and the electrical signals are quantised. In this case, monitoring the voltage corresponds to a continuous quantum measurement on the circuit. The nature of continuous measurements in quantum systems has two direct consequences for the circuit dynamics, namely back-action and stochastic evolution.

The first consequence, the back action, consists in the unavoidable perturbation of the state of the circuit by the observer performing the continuous measurement. However, this seems a necessary consequence of coupling the quantum circuit to a memristor, because otherwise, a memristor could be used to engineer a non-perturbative measurement device. Let us consider a memristor with a very high resistance value coupled in parallel
to a circuit. In this case, almost no current flows through the memristor branch and, in a small time interval, damping and decoherence in the circuit could be neglected. Still, the state variable changes and an observer could perform a voltage measurement without perturbing the circuit by reading out this change.

The second consequence, stochastic dynamics, is a direct result of the intrinsic probabilistic nature of the quantum measurement continuously performed during the evolution. This produces different quantum trajectories in every realisation of the experiment, which result in a probability distribution of the state variable. Consequently, the resistance value also differs in every run of the experiment. Again, this seems reasonable for a physical memristor, since only a probability distribution of the classical state variable can take account of the surplus of information contained in the superposition of circuit states.

In this sense, the feedback model promotes a quantisation method for a complex dissipative classical system with memory like the memristor. First, it involves measurement and its peculiarities in quantum mechanics. Second, it needs to couple the circuit to a bath to enable dissipation. Third, it requires a feedback step controlling this coupling via the record of measurement results. Therefore, the feedback model comprises two baths, the measurement bath and the dissipation bath, connected via a classical feedback line which models their interaction.

Usually, in the previous literature, feedback is regarded as a control tool. However, in this thesis, it plays a different role, simulating the physical mechanism which relates state variable and resistance, and thus generates the memory effects in the circuit.

3.2 Quantum memristor master equation

In this section, the aforementioned feedback model of the memristor is translated into a master equation. First, it is explained how the tuning of the the coupling coefficients in the Devoret model provides a tuneable damping. Second, a conditioned evolution is proposed in the case for which the tuneable damping is a function of measurement records. We call this the conditioned quantum memristor master equation. Afterwards, the form of the unconditioned evolution is discussed, and how its memory effects are related to quantum non-Markovianity is analysed.

Tuneable damping

If the state variable is a constant, the memristor is a linear resistor. The effect of a linear resistance in parallel to a circuit is mimicked by a linear coupling to a bath of harmonic
oscillators with an Ohmic spectral density, whose main parameter is the damping rate $\gamma$, see Section 2.1.2.2. By means of this parameter, which summarises the coupling coefficients of the bosonic modes to the circuit, the value of the resistance is chosen,

$$H_I(R) = \varphi \sum_k \kappa_k(R) \varphi_k \leftrightarrow J_{\gamma(R)}(\omega) \leftrightarrow R.$$  

The relation between the damping rate and the resistance value was explained in Section 2.2.1.2 and yielded

$$\gamma(R) = \frac{1}{2RC_{tot}},$$

where $C_{tot}$ is the reciprocal sum of all the capacitors connected to the top node of the circuit, which the memristor shunts. Moreover, the voltage across the memristor reads

$$V = \frac{q}{C_{tot}}.$$  

Therefore, monitoring the voltage corresponds to monitoring the node charge. As discussed in Section 2.2.2, the output of a continuous charge measurement is given by the mean value of the charge and white noise whose amplitude depends on the measurement strength $k$,

$$M_q(t) = \langle q(t) \rangle + \frac{1}{\sqrt{8k}} \zeta(t).$$  (3.1)

By taking this measurement outcome as an estimation of the voltage across the memristor, the update of the state variable in the second memristor Equation 2.160 yields

$$\dot{\mu} = f(M_q(t)).$$  (3.2)

Note that this an implicit SDE and thus, the conversion to an explicit increment $d\mu$ might be involved (see Appendix A).

The increment $d\mu$ determines the update of the resistance to the value $R(\mu + d\mu)$. A corresponding update of the damping parameter in the spectral density $J(\omega)$ implements this change of the resistance. Thus, the feedback step has to alter the interaction Hamiltonian between circuit and bath, such that the spectral density is shifted as depicted in Figure 3.2.

**Conditioned quantum memristor equation**

On the level of the circuit density matrix, the measurement of a circuit variable and the feedback step on the joint system, circuit and bath, lead to dynamics for the conditioned evolution, split into three terms,

$$d\rho = d\rho_H + d\rho_{\gamma(\mu)} + d\rho_{\text{meas}}.$$  (3.3)
Figure 3.2: After calculating the update of the state variable $d\mu$ the bath-circuit interaction has to be altered to produce an overall shift of the Ohmic spectral density. The updated shape has to correspond to the new damping rate $\gamma(\mu + d\mu)$ in order to adapt to the change of the memristor resistance.

The increment $d\rho_{\text{meas}}$ due to the continuous measurement has a stochastic part. In principle, the damping term $d\rho_{\text{damp}}^{\gamma(\mu)}$ could also involve noise due to the stochasticity of its input. Furthermore, as the feedback controls the circuit-bath interaction term, it is possible that the Hamiltonian part $d\rho_H$ changes e.g. due to a Lamb shift in the circuit frequency. Putting aside the discussion of these terms to a later section, it is assumed that the three parts have the following form,

\[
\begin{align*}
    d\rho_H &= -\frac{i}{\hbar} [H_{\text{circ}}, \rho(t)]\, dt, \\
    d\rho_{\text{damp}}^{\gamma(\mu)} &= -\frac{i\gamma(\mu)}{\hbar} [\varphi, \{q, \rho(t)\}]\, dt - \frac{2Ck_BT\gamma(\mu)}{\hbar^2} [\varphi, [\varphi, \rho(t)]]\, dt, \\
    d\rho_{\text{meas}} &= -k [q, [q, \rho(t)]]\, dt + \sqrt{2k} (\{q, \rho(t)\} - 2 \langle q \rangle \rho(t))\, dW.
\end{align*}
\]

The damping part has a Caldeira-Leggett form with the state-variable-dependent damping rate $\gamma(\mu)$. One realisation of the corresponding quantum trajectory is generated by these three increments and by the update of the state variable due to the measurement output, Equation 3.2. Hence, the conditioned evolution is given by a set of two coupled stochastic differential equations.

The proposed structure of the conditioned evolution is justified by some requirements of the involved time scales. The bath is assumed to fulfil the Born-Markov approximation with respect to both the system dynamics and the feedback control. This translates into a bath thermalisation time scale much faster than the control time of the tuning of the interaction Hamiltonian. Thus, excitations introduced by the control distribute sufficiently fast in the bath and it can be assumed to stay in a thermal state. Furthermore, the exchange rate of excitations between bath and system should be slower than the control speed. Otherwise, the assumption about the Caldeira-Leggett structure for the
Figure 3.3: The arrow visualises the order of the time scales in the measurement+feedback model of the memristor that underlie the phenomenological quantum memristor master equation. The red triangle indicates the coarse graining assumption of the time evolution. In smaller time intervals the system’s evolution is not resolved and involves the immediate thermalisation of the bath, the control time during which the interaction Hamiltonian is updated and the exchange of bath-system excitations. The system evolution, damping and the measurement are slower processes.

damping term would not hold. A tentative ordering is outlined in Figure 3.3, emphasising that the phenomenological master equation presents a coarse grain evolution.

The form of the conditioned quantum memristor master equation seems Markovian, because the increments involve the density operator explicitly only at time $t$. Yet, the state variable contains partial information about the history of the density operator, what is shown by rewriting

$$
\mu(t) = \int_{t_0}^t ds f \left( \text{Tr} [q \rho(s)] + \frac{1}{\sqrt{8k}} \zeta(s) \right).
$$

(3.7)

Since $\mu$ is the control parameter for the damping rate $\gamma(\mu)$, the conditioned evolution is non-linear in $\rho$. This non-linearity appears non-locally in time, generating a non-Markovian form. Both features are expected in the conditioned feedback evolution, see Section 2.2.3. The Wiener process enters into the value of $\mu$ in two ways, directly via the noise term $\frac{1}{\sqrt{8k}}$, and in form of the conditioned state $\rho(s)$ due to the back action.

**Evolution of the ensemble average**

The unconditioned density operator $\bar{\rho}$ of the circuit corresponds to the average over the ensemble of all possible conditioned trajectories, see for example Section 2.2.2.1,

$$
\bar{\rho} := \langle \langle \rho \rangle \rangle.
$$
The increments of the unconditioned evolution read

\[
\begin{align*}
\mathrm{d}\bar{\rho}_H &= -\frac{i}{\hbar} [H_{\text{circ}}, \bar{\rho}(t)] \, \mathrm{d}t, \\
\mathrm{d}\bar{\rho}_{\text{damp}} &= -\frac{i}{\hbar} [\varphi, \{q, \langle \gamma(\mu)\rho(t) \rangle \}] \, \mathrm{d}t - \frac{2CK_BT}{\hbar^2} [\varphi, [\varphi, \langle \gamma(\mu)\rho(t) \rangle]] \, \mathrm{d}t, \\
\mathrm{d}\bar{\rho}_{\text{meas}} &= -k [q, [q, \bar{\rho}(t)]] \, \mathrm{d}t.
\end{align*}
\]

(3.8) (3.9) (3.10)

Due to the Ito property, \( \langle \langle \rho \mathrm{d}W \rangle \rangle = 0 \), the non-linear part in the measurement contribution to the evolution is removed. The interesting object is the expectation value of the product of the damping constant and the density operator. Both are generated by the same stochastic process, so that in general \( \langle \langle \gamma(\mu)\rho(t) \rangle \rangle \neq \langle \langle \gamma(\mu) \rangle \rangle \bar{\rho} \). Therefore, a closed expression for the evolution of \( \bar{\rho}(t) \) cannot be written. This situation is due to the non-linear and non-Markovian form of the conditioned evolution, a setting commented on in Section 2.2.3. Still, the ensemble average defines a non-linear map performing the time evolution of the unconditioned state. It has to depend on the initial state variable value \( \mu(0) \),

\[ \bar{\rho}(t) = \langle \langle \rho(t) \rangle \rangle =: \Phi^{\mu(0)}_t(\bar{\rho}(0)) \]

As this does not constitute a linear map, the discussion in Section 2.2 about quantum non-Markovianity cannot be applied directly. But from the above equations, one might at least point to conditions that would make it a linear quantum Markovian evolution. If the expectation value factorises

\[ \langle \langle \gamma(\mu(t))\rho(t) \rangle \rangle = \langle \langle \gamma(\mu) \rangle \rangle \bar{\rho}(t) \]

and \( \langle \langle \gamma(\mu(t)) \rangle \rangle \equiv \text{const} \) becomes a constant, the update is of an approximate Lindblad form, see Section 2.1.2.2, and thus Markovian.

The meaning of the unconditioned and conditioned evolution can be illustrated by the external and internal observer from the discussion of the feedback model of the memristor. Their access is schematically depicted in Figure 3.4. If the external observer wants to obtain the statistics of an observable of the circuit, he needs to reinitialise the whole system various times. Reinitialisation implies to prepare the same circuit state and to couple it to an identical memristor with the same initial state variable. Afterwards, he has no insight into the compound environment and thus, he is averaging over the whole ensemble of quantum trajectories. The internal observer has access to the measurement results and, in principle, is able to post select the same measurement history. In this way, he could perform a tomography of the conditioned density matrix. Furthermore, one might consider more complicated conditioned states. For example, the state variable i.e. the damping is known at a certain time and thus, the averaging only includes those trajectories which lead to the corresponding state variable value.
Figure 3.4: The scheme shows the elements for the microscopic feedback model underlying the quantum memristor master equation. The system/circuit is in contact with a bipartite environment consisting of a measurement-control device and a Markovian bath. In order to change the dissipation rate in the system, the coupling with the bath is manipulated. The manipulation depends on the state variable of the control which is altered as a function of the measurement record. The internal observer (I) follows the conditioned evolution whereas the external observer ignores the measurement output and has only access to the unconditioned evolution.

3.3 Road map for a microscopic feedback model

In the previous section, a feedback model was taken as an inspiration to propose the conditioned quantum memristor master equation. Although we have provided heuristic arguments to justify its form, it would be desirable to make those arguments rigorous by providing a microscopic derivation including the bath and the explicit form of the feedback. The following section sketches a small road map of this derivation. It poses a problem in the frame of measurement-based quantum feedback that, to our knowledge, has not been treated so far.

Formulating the model of the quantum memristor in the measurement-based feedback scheme, the system which is feedback controlled consists of the circuit and the bath of harmonic oscillators. Finally, the system of interest is the circuit, so that the bath has to be traced out for the final evolution. This is the new aspect of the problem, an open quantum system for which the dynamics of the full system is generated by feedback.

First, the problem of providing this full system dynamics has to be solved. The starting point is the conditioned evolution of the joint system of circuit and bath, uncoupled, for which some circuit variable $c \otimes 1$ is monitored. The conditioned evolution reads

$$\frac{d\rho_M}{dt} = -\frac{i}{\hbar}[H_{\text{circ}} + H_B, \rho_M] + 2kD[c \otimes 1]\rho_M(t)dt + \sqrt{2k}\mathcal{H}[c \otimes 1]\rho_M(t)dW.$$  (3.12)
Next, a feedback term according to Equation 2.156 has to be engineered. The explicit form of the feedback increment should involve a Hamiltonian part governing the interaction,

\[-\frac{i}{\hbar} [H_I [\mu(0), M(s); 0 \leq s < t], \rho_M].\]

This interaction Hamiltonian is a functional of the measurement records and the initial value of the state variable \(\mu\). The functional has to depend on both the state variable dynamics and the resistance function \(R(\mu)\). Therefore, it includes all previous measurement results. Precisely, it should have the form

\[H_I = \sqrt{\gamma(\mu)} \sum \hat{\kappa}_k \phi \varphi_k. \quad (3.13)\]

The coupling coefficients \(\hat{\kappa}_k\) provide an Ohmic spectral density with \(\gamma = 1\), such that the above interaction Hamiltonian corresponds to a spectral density with \(\gamma = \gamma(\mu)\). The challenge is to characterise the noise properties of the feedback term, in which the measurement records enter through both the possible non-linear state variable dynamics and the definitely non-linear resistance function (as \(R > 0\)). Furthermore, remembering the derivation of the Caldeira-Leggett master equation, it might be necessary to include also a counter term Hamiltonian \(H_c\) in the feedback to provide reasonable dynamics.

Depending on this noise properties, further terms besides the Hamiltonian part, will appear in the final conditioned evolution of the full system. In order to get a tractable master equation for the circuit, one could try to use the projection operator technique discussed in Section 2.2.1. Differently from its conventional application framework, here, the relevant and irrelevant part will be stochastic variables, and could already include terms non-local in time due to the non-Markovian feedback. It is a compelling question, if this still allows to go along a treatment like in the derivation of the Caldeira-Leggett master equation, and whether one can reproduce the phenomenological form of the quantum memristor master equation proposed in this chapter.
Chapter 4

A quantum LC circuit coupled to a memristor

In a classical circuit, a memristor introduces a history-dependent damping. Does this memory effect survive in the quantum regime and if so, how is it altered? In the previous chapter, a phenomenological quantum master equation for a circuit in contact with a memristor was introduced. The structure of the unconditioned evolution, followed by an external observer with no information about the memristor, reveals that the essential object to study memory effects is the correlation of the tuneable damping rate and the conditioned density matrix

$$\langle \langle \gamma(\mu(t))\rho(t) \rangle \rangle.$$

In general, this should be a complex object, hard to characterise analytically. Below a numerical study of this object is presented for the case of Gaussian states in a LC circuit coupled in parallel to a voltage controlled memristor. These states are fully characterised by their mean and variances and preserve their form under the conditioned evolution. Hence, they allow a simple numerical treatment of the evolution via a closed set of coupled stochastic differential equations. The observables of the unconditioned evolution can then be studied via an ensemble of trajectories.

First, the equations of the five moments are presented, and it is explained how the external observer could measure the hysteresis of the memristor. Furthermore, it is discussed why the hysteresis is the right feature to look for non-Markovianity in the unconditioned evolution. Second, the numerical case study provides a parameter set, in which the hysteresis survives during the unconditioned evolution, showing that coupling the quantum circuit to the memristor indeed induces memory. However, the conditions on measurement strength and initial circuit preparation, which seem necessary to generate a history-dependent evolution, constitute a regime that leads to classicalisation of the
Chapter 4. Memristor in a quantum LC circuit

4.1 Dynamics of Gaussian states

The following section introduces the case study in three steps. In the first part, the set of stochastic differential equations governing the evolution of the cumulants of a Gaussian state is presented. In the second part, the set of equations is completed by fixing a linear form for the state variable dynamics. Finally, the reason why the hysteresis offers a good indicator for the presence of memory in the unconditioned evolution is discussed.

The circuit, which we are considering in this section, is depicted in Figure 4.1. For the numerical study, the same unitless variables employed in the discussion of the Caldeira-Leggett master equation are used for convenience. The dimensionless flux and charge operators are defined as

\[ \varphi \rightarrow \frac{\hat{\varphi}}{\varphi_0}; \quad \varphi_0 = \sqrt{\frac{\hbar}{\omega C}} \]

\[ q \rightarrow \frac{\hat{q}}{q_0}; \quad q_0 = \sqrt{\hbar \omega C} \]

where \( \omega \) is the frequency of the LC circuit and \( C \) is the capacitance. Furthermore, the thermal frequency is introduced

\[ \lambda = \frac{k_B T}{\hbar}, \]
as well as the projection frequency (for a discussion of this quantity see Section 2.2.2.1)

\[ \tau = kq_0^2. \]

Time is rescaled by the circuit frequency \( t \to \omega t \), i.e. \( t = 2\pi \) corresponds to one period and all the frequencies are understood as multiples of \( \omega \).

**Conditioned evolution of the cumulants**

The conditioned evolution from Equation 3.4, 3.5 and 3.6, rewritten in terms of those two frequencies and the damping rate function \( \gamma(\mu) \), reads

\[
\text{d}\rho = \text{d}\rho_H + \text{d}\rho_{\text{damp}} + \text{d}\rho_{\text{meas}}
\]

with

\[
\text{d}\rho_H = -\frac{i}{2} [q^2 + \varphi^2, \rho] \, dt, \quad (4.1)
\]

\[
\text{d}\rho_{\text{damp}} = -i\gamma(\mu) [\varphi, \{q, \rho\}] \, dt - 2\gamma(\mu)\lambda [\varphi, [\varphi, \rho]] \, dt, \quad (4.2)
\]

\[
\text{d}\rho_{\text{meas}} = -\tau [q, [q, \rho]] \, dt + \sqrt{2\tau} (\{q, \rho\} - 2 \langle q \rangle \rho) dW. \quad (4.3)
\]

The increment of the mean value of an operator \( A \) has the form

\[
\text{d}\langle A \rangle = \text{Tr} \left[ A (\text{d}\rho_H + \text{d}\rho_{\text{damp}} + \text{d}\rho_{\text{meas}}) \right]
\]

\[
= -\frac{i}{2} \langle [A, q^2 + \varphi^2] \rangle \, dt
\]

\[
- i\gamma(\mu) \langle [q, [A, \varphi]] \rangle \, dt - 2\gamma(\mu)\lambda \langle [\varphi, [\varphi, A]] \rangle \, dt
\]

\[
- \tau \langle [q, [q, A]] \rangle \, dt + \sqrt{2\tau} (\{q, A\} - 2 \langle q \rangle \langle A \rangle) dW. \quad (4.4)
\]

Calculating the updates of the cumulants of an initially Gaussian state yields a closed set of stochastic differential equations, see Section 2.2.2.3 on continuous measurement. Hence, the Gaussian structure of the state is preserved during the conditioned evolution.

The evolution of the cumulants is given by

\[
\text{d}\langle \varphi \rangle = \langle q \rangle \, dt + \sqrt{8\tau} C_{\varphi,q} dW, \quad (4.5)
\]

\[
\text{d}\langle q \rangle = -\langle \varphi \rangle \, dt - 2\gamma(\mu) \langle q \rangle \, dt + \sqrt{8\tau} V_q dW, \quad (4.6)
\]

\[
\text{d}V_{\varphi} = 2C_{\varphi,q} dt + 2\tau (1 - 4C_{\varphi,q}^2) dt, \quad (4.7)
\]

\[
\text{d}V_q = -2C_{\varphi,q} dt - 4\gamma(\mu) (V_q - \lambda) \, dt - 8\tau V_q^2 dt, \quad (4.8)
\]

\[
\text{d}C_{\varphi,q} = (V_q - V_{\varphi}) dt - C_{\varphi,q} (2\gamma(\mu) + 8\tau V_q) dt. \quad (4.9)
\]
Table 4.1: Overview of the frequencies governing the conditioned evolution of a Gaussian state in the LC circuit coupled to a memristor. In the text, they appear in units of the circuit frequency $\omega$.

<table>
<thead>
<tr>
<th>$\gamma(\mu)$</th>
<th>$\lambda$</th>
<th>$\tau$</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>damping rate</td>
<td>thermal frequency</td>
<td>projection frequency</td>
<td>memory frequency</td>
</tr>
<tr>
<td>$\gamma(\mu) = \frac{G(\mu)}{2C}$</td>
<td>$\lambda = \frac{kqT}{h}$</td>
<td>$\tau = kq_0^2$</td>
<td>$\nu = \frac{q_0}{\sqrt{C\Phi_0}}$</td>
</tr>
</tbody>
</table>

In contrast to the case of solely monitoring the charge, the evolution of the variances is not deterministic. There is no explicit noise dependence in form of a Wiener increment, but the seed of stochasticity is the damping rate $\gamma(\mu)$, which is partially controlled by noise.

**State variable dynamics**

Furthermore, the function $f(M_q(t))$ providing the state variable dynamics in Equation 3.2 has to be specified. In particular, the noise structure in the function output has to be understood, in order to transform the implicit equation into an explicit increment. Despite the convenient properties of the Wiener noise in the input, this constitutes a difficult task for a general $f$, which mostly relies on linearisation [52]. Hence, here a linear function is considered,

$$f(V(t)) = \frac{V(t)}{\Phi_0}.$$  

(4.10)

$\Phi_0$ is the generalised flux that has to flow through the memristor to change the state variable $\mu$ significantly. In the case of the memristive contribution in the Josephson junction, see Section 2.3.1, this corresponds to the flux quantum. The linear structure readily enables to translate the implicit Equation 3.2 into an explicit increment, which yields by using $M_V(t) = \frac{M_q(t)}{C}$ and the unitless time and frequency,

$$d\mu(t) = \frac{q_0}{C\Phi_0\omega} \left( \langle q(t) \rangle dt + \frac{dW}{\sqrt{8\tau}} \right)$$

$$= \nu \left( \langle q(t) \rangle dt + \frac{dW}{\sqrt{8\tau}} \right).$$  

(4.11)

The frequency $\nu$ defines the rate of change of the state variable per unit of vacuum fluctuation charge ($\times \sqrt{2}$) on the capacitor. Hence, it determines the importance of the corresponding voltage signal for the memory update. All the frequencies governing the conditioned evolution are summarised in Table 4.1.

These coupled non-linear SDEs are written in the Ito form, which enables a direct implementation of the explicit Euler algorithm to numerically solve them (see Appendix A for details). Consequently, the observables of the unconditioned density operator $\bar{\rho}$
are calculated from an ensemble generated by different Wiener noise trajectories. For an ensemble of conditioned density operators \( \{\rho_j\}_{1 \leq j \leq N} \) of size \( N \), the unconditioned mean of the observable \( A \) at time \( t \) is estimated via \cite[p. 423]{12} 

\[
\langle\langle A \rangle\rangle = \text{Tr} (A \rho(t)) = \langle\langle \text{Tr} (A \rho(t)) \rangle\rangle 
\approx \frac{1}{N} \sum_j \langle A \rangle_j (t)
\]  

(4.12)

where the linearity of the trace was used in the second step. The error of the estimate is

\[
\delta\langle\langle A \rangle\rangle = \sqrt{\frac{\frac{1}{N} \sum_j \langle A \rangle_j^2 (t) - \left( \frac{1}{N} \sum_j \langle A \rangle_j (t) \right)^2}{N}}.
\]

(4.13)

The ensemble also probes the probability distribution of \( \mu \) and \( \gamma(\mu) \).

**Hysteresis and Non-Markovianity**

The striking memory footprint of the classical memristor is its I-V-hysteresis curve. Although the external observer cannot directly measure the current through the memristor, he is still able to infer the current loss into the memristor branch. The unconditioned evolution of the charge is equal to the average of Equation 4.6 and reads

\[
\frac{d}{dt} \langle\langle q \rangle\rangle(t) = -\langle\langle \varphi \rangle\rangle(t) - 2\langle\langle \gamma(\mu) \langle q \rangle \rangle \rangle(t),
\]

(4.14)

representing the conservation of charge at the top node in Figure 4.1. The charge flowing from or onto the plates of the capacitor has to either flow through the inductor or the memristor branch. Hence, by measuring the evolution of the mean charge and the mean flux, the observer can tell the amount of current through the memristor. Furthermore, the mean voltage is proportional to the mean charge on the capacitor and thus, by plotting \( \langle\langle q \rangle\rangle - 2\langle\langle \gamma(\mu) \langle q \rangle \rangle \rangle \), he obtains the hysteresis of the memristor.

Since in the conditioned evolution charge is not smooth in time due to the Wiener noise back action, the current is not well defined in any of the branches. The active nature of the quantum measurement introduces or leaks out energy, which results in instantaneous fluctuations of the charge on the capacitor. Still, if the state variable is known, the expected current through the memristor branch can be deduced by

\[
\langle I_M \rangle = G(\mu) \langle V \rangle = 2\gamma(\mu) \langle q \rangle.
\]
The hysteresis is a sufficient condition for non-Markovianity in the unconditioned evolution, and intuitively, should also be a necessary requirement. This means that, if the ensemble average $\langle\langle \gamma(\mu) \langle q \rangle \rangle\rangle$ factorises and $\gamma(\mu)$ is constant, the hysteresis vanishes. From this point on, the update $d\tilde{\rho}$ is Markovian, except a potential dependence of the damping constant on the initial conditions. Thus, exploring the regimes for which the hysteresis is emphasised or suppressed permits to identify the cases where non-Markovianity plays a role. Whether this memory effect can also be related to the concept of quantum non-Markovianity, needs a thorough discussion of non-Markovian quantum feedback, which lacks in the literature and goes beyond the scope of this thesis.

4.2 Case study of hysteresis in a quantum memristor

The full parameter space of the problem comprises not only the four frequencies of Table 4.1, but also the damping function, and the initial conditions for the state variable and for the cumulants of the Gaussian state. Therefore, in order to identify the regimes leading to memory effects, in this section, we present a case study for the memristive term in the Josephson junction. Its conductance function reads

$$G(\mu) = G_0(1 + \epsilon \cos \mu)$$

$$\Rightarrow \gamma(\mu) = \gamma_0(1 + \epsilon \cos \mu)$$

(4.15)

with $0 < ||\epsilon|| < 1$ and $\gamma_0 = \frac{G_0}{2\tau}$. Let us remark that, for $\epsilon = 0$, the feedback is turned off and the model reduces to a voltmeter and a constant resistor.

The minimal requirement for a visible memory effect, i.e. hysteresis, is a significant change of the conductance, i.e. the damping. This change has its origin in the evolution of the state variable, which should be controlled by relevant information. Hence, this change is mainly due to the charge signal instead of the measurement noise. The measurement noise enters primarily through the projection frequency $\tau$, but in two different ways. On the one hand, a large $\tau$ decreases the contribution of the Wiener increment in the update of the state variable making the information about the currently measured value of the charge more reliable. On the other hand, this high rate of information gain has a big impact on the system state, and abruptly changes the first moments in Equation 4.5 and Equation 4.5. Furthermore, the amplitude of this change depends on the variances and is suppressed for small $\tau$.

In pursuance of identifying a regime of the projection frequency $\tau$ suppressing sufficiently both contributions, it is helpful to consider a deterministic variance evolution with a fixed value of the damping rate $\gamma$. In this case, the stationary values for covariance and for
charge variance have the following dependence on $\tau$,

$$
C_{\varphi,q}^{st} = -\frac{1}{8\tau} \left( \sqrt{1 + (4\tau)^2} - 1 \right),
$$

(4.16)

$$
V_q^{st}(\tau) = \frac{\sqrt{\gamma^2 + 4\tau(2\gamma\lambda - C_{\varphi,q})} - \gamma}{4\tau}.
$$

(4.17)

Hence, the total diffusion is approximately minimised, if the sum of the noise terms in the first moments evolution, given by Equation 4.5 and Equation 4.6, and in the state variable evolution, given by Equation 4.11, is small,

$$
D(\tau) = \sqrt{8\tau} V_q(\tau) + \sqrt{8\tau}|C_{\varphi,q}| + \frac{1}{\sqrt{8\tau}}.
$$

(4.18)

For large $\tau$, the covariance contribution prevails and the noise grows with $\sqrt{\tau}$. In this limit, the charge variance contribution is constant. In the small $\tau$ regime, the noise in the state variable $\frac{1}{\sqrt{\tau}}$ dominates. All the contributions are shown in Figure 4.2 and the minimum of $D(\tau)$ gives an idea where the total noise in the system is suppressed.

Furthermore, the initial state preparation and the choices of memory frequency, damping rate and temperature have to be fixed. In the following, coherent states will be considered, i.e. $V_q(0) = 0.5, V_\varphi = 0.5, C_{\varphi,q} = 0$. This choice is rather arbitrary, because the variances very quickly approach their quasi-stationary values, up to small changes due to the varying damping rate. Moreover, the product of memory frequency and the typical charge on the capacitor, $\nu \langle q_t \rangle$, determines the change of the state variable during half a period of the LC is oscillation and thus, the emphasis of the hysteresis. Initial conditions are chosen such that $1 \leq \nu \langle q_t \rangle \leq \pi$, avoiding multiple crossings. The damping continuously decreases the maximum charge value and consequently, the hysteresis survives longer, if the damping is small. The temperature appears in the noise contributions via
the charge variance and thus, should also be chosen small, but sufficiently large to provide
the high temperature limit, $\lambda \gg 1$, required by the Caldeira-Leggett approximation.

**Observing the hysteresis**

In the first part of this section, we compare the conditioned evolution for two projection
frequencies distinct from the optimal one, showing how the noise due to the measurement
appears in the hysteresis. Figure 4.6 and Figure 4.7\(^1\) show illustrative realisations for
a very fast and a very slow measurement. In the case of a fast measurement ($\tau \geq 1$),
the noise in the evolution of the state variable is highly suppressed. But both mean
charge and mean flux fluctuate, and the measurement introduces considerable energy
fluctuations. In the case of a slow measurement ($\tau \ll 1$), it is the other way around.
The state variable highly fluctuates and thus, the damping rate also becomes very noisy.
However, mean charge and mean flux evolve approximately unperturbed, and the energy
decays to the equilibrium value.

The different noise sources can be distinguished in the hysteresis. In Figure 4.3, we
show a comparison between the hysteresis in the conditioned evolution and the one in
the corresponding classical noiseless circuit, obtained by plotting $q(t)$ vs. $2\gamma(\mu)q(t)$ as
solutions of

$$
\frac{d}{dt} q(t) = -\varphi(t) - 2\gamma(\mu(t))q(t),
\frac{d}{dt} \varphi(t) = q(t),
\frac{d}{dt} \mu(t) = \nu q(t).
$$

The slow measurement case shows mainly fluctuations along the memristor current axes
(vertical) due to the very noisy damping rate. On the other hand, in the fast measurement
case, the fluctuations are more pronounced along the charge axes (horizontal), reflecting
the increased back action on the first moments.

The discussion of the diffusion contributions is very well illustrated in the evolution of
the damping rates. Figure 4.8 portrays the diffusion of the damping rate by sampling
different instances of its evolution for the two values of $\tau$ considered so far. In both cases
the damping rates belonging to different trajectories progressively dephase such that
the probability distribution of damping rate values diffuses. Finally, it is distributed
uniformly over the possible range and the ensemble mean reaches the stationary value
$\gamma_{st} = \frac{\gamma(0) + \gamma(\pi)}{2}$. The origin of the dephasing is very different in the two cases and
highlights how the memory effects can be corrupted.

\(^1\)In order to support readability, some of the figures are put at the end of the chapter, starting at
page 84.
Figure 4.3: Comparison of individual hysteresis curves in the slow measurement (left) and fast measurement case (right). The memristive current is $I_M = 2\gamma(\mu(t)) \langle q \rangle$. The colouring of the line depicts the time evolution. The dashed black line in the background represents the hysteresis in the equivalent circuit with classical noiseless signals.

If the measurement is slow, the circuit evolves nearly unperturbed. All the trajectories share almost the same voltage history. Still, the information about this voltage history recorded in the state variable is very unreliable due to the large measurement noise. In every trajectory, the history encoded in the state variable fluctuates wildly around the actual one and thus, the damping rates dephase. In the fast measurement case, the individual damping rates are almost noiseless, but they correspond to different voltage histories. The stochastic back action heavily affects the evolution of the circuit variables, and opens multiple voltage history branches. To sum up, there are two extremes: the slow measurement extreme of one history, but faulty records, and the fast measurement extreme of perfectly recorded multiple histories. The optimal case $\tau \approx 0.2$ constitutes a balance between these extremes.

Now, let us discuss how this dephasing of the damping rate corrupts the memory effect in the unconditioned evolution. In Figure 4.4, the hysteresis in this case is shown for four values of $\tau$. In all four curves, the hysteresis is clearly visible during the first oscillation. However, for longer times, it is significantly suppressed for both very slow and very fast measurements, effectively collapsing to a line. This collapse is due to the aforementioned diffusive processes in the first moments and in the state variable. For the optimal choice of $\tau$, these are suppressed, and the hysteresis coincides almost with the classical one.

The presence of the hysteresis clearly shows that the quantum memristor model has successfully transferred the memory feature of the memristor to the evolution of the
Before addressing this issue, let us discuss a peculiarity of the quantum memristor hysteresis. So far, only the evolution of the mean damping rate has been discussed, while the potential correlation with the average charge has been ignored. Due to the special form of the non-Markovian term in the master equation, the hysteresis is not necessarily pinched at the origin. The evolution of the mean damping rate is sufficient to understand the collapsing hysteresis because, for the values of $\tau$ and the population of the circuit state. However, the question, whether quantum features are preserved in the state of the circuit, still remains.

Figure 4.4: Comparison of the hysteresis in the unconditioned evolution with the case of a classical noise free circuit. The ensemble average leads to a final collapse of the hysteresis in the very slow and very fast measurement cases (A), (B) and (D). The optimal value of $\tau$ is depicted in (C) where the classical hysteresis curve is almost indistinguishable from the one produced in the mean. Estimation errors due to the finite ensemble size ($N=3000$) are omitted for visibility.
considered above, the absolute value of the deviation from factorisation,

$$|\delta_q| = |\langle\langle \gamma(\mu)q \rangle \rangle - \langle\langle \gamma(\mu) \rangle \rangle \langle\langle q \rangle \rangle|,$$

(4.19)
is found to be much smaller than the product $|\langle\langle \gamma(\mu) \rangle \rangle||\langle\langle q \rangle \rangle|$; in particular, this guarantees

$$\langle\langle \gamma(\mu)q \rangle \rangle = \left(1 + \frac{\delta_q}{\langle\langle \gamma(\mu) \rangle \rangle \langle\langle q \rangle \rangle} \right) \langle\langle \gamma(\mu) \rangle \rangle \langle\langle q \rangle \rangle \approx \langle\langle \gamma(\mu) \rangle \rangle \langle\langle q \rangle \rangle.$$

In Figure 4.9, the deviation $\delta_q$ in the cases $\tau = 0.005$ and $\tau = 0.2$ is compared. The simulations show that the deviation is small, especially for the noise suppressed case $\tau = 0.2$. However, the deviation has an interesting effect which is clear by rewriting

$$\langle\langle \gamma(\mu)q \rangle \rangle(t) = \langle\langle \gamma(\mu) \rangle \rangle(t) \langle\langle q \rangle \rangle(t) + \delta_q(t).$$

Hence, the external observer detects a current through the memristor branch, even though the mean value of the voltage, i.e. the charge on the capacitor, is zero. The $I$-$V$ curve of the memristor in the unconditioned evolution is not necessarily pinched at the origin. This is only the case, if the zeros of $\langle\langle q \rangle \rangle$ and the zeros of $\delta_q(t)$ coincide in time. Regarding the $\tau = 0.005$ case, the zero crossings of the mean charge correspond to non-zero values of the deviation. A zoom into the region around the origin of Figure 4.4a shows the deviation from the pinched hysteresis loop, as depicted in Figure 4.10.

The shift of the pinched point might raise doubts about the actual passivity of the quantum memristor. As the $I$-$V$ curve crosses the second and fourth quadrant for a small amount of time, the quantum memristor provides power instead of consuming it. But we believe that this corresponds to an inductive, respectively capacitive, contribution generated by the capability of the aggregate of measurement device, feedback loop and tuneable bath to store and leak small amounts of energy.

A comprehensive analysis of this contribution would need a numerical study of a larger range of parameters or, preferably, an analytical understanding of the correlation, to our knowledge an open problem in stochastic processes. The simulations indicate that the amplitude of $\delta_q$ is minimal around the optimal $\tau$ and thus, for the primary interest of the memory features, it plays a minor role.

4.3 Memory and the quantum-to-classical transition

In the case study above, it has been shown that the hysteresis in the classical and in the quantum case almost coincide for the right choice of the projection frequency $\tau$. The hysteresis marks the non-Markovianity of the unconditioned master equation. Therefore,
a circuit, which isolated evolves unitarily i.e. in a Markovian way, acquires memory, in the sense of a non-Markovian evolution, when coupling it to the memristor.

Still, the unitary evolution assured the preservation of quantum characteristics, like coherent superpositions. Can these features survive when coupled to a memristive environment? In the same manner, usual open quantum systems successively decohere, the memristive environment affects the quantum nature of the state. Decoherence is a process that suppresses interference effects and thus, turns the quasi-probability quantum Wigner function into a probability distribution in phase space [13, p. 1]. This is a part of the quantum-to-classical transition that is present in most open quantum systems.

How is decoherence related to the memory effects in the quantum memristor master equation? The resistance of the memristor introduces a thermal broadening term due to the fluctuation-dissipation theorem. As the Caldeira-Leggett regime corresponds to a high temperature limit, it is the main source of decoherence in the above model. Still, also for low temperatures, the feedback model contains decoherence due to the measurement.

It is considered that continuous measurements play a role in the quantum-to-classical transition due to the emergence of 'localised' trajectories from quantum dynamics [13, 28, 38]. In the following section, we argue that the conditions on the projection frequency for the appearance of the hysteresis and this kind of classicalisation coincide. Loosely speaking, a localised trajectory keeps the state variable localised as well and thus, allows for memory effects.

**Conditions on \( \tau \) to provide hysteresis**

In the previous discussion of the noise terms, the projection frequency minimising their total contribution was determined. However, it is not the total amplitude of the noise what matters, but its relative contribution with respect to the signal i.e. the value of the first moments. Once they are comparable, so the deviation from the mean is of the same order as the mean, the state variable contains as much noise as relevant signal and hence, effectively forgets the past trajectory of the circuit.

A Wiener noise contribution of the form \( DdW \) induces a linear growth in time of the variance \( \propto D^2t \). Hence, if up to a time \( T \), this noise were to be small in comparison with the typical value of the first charge moment \( q_t \), the following two conditions would have to be fulfilled. First, the means should evolve sufficiently unperturbed, i.e. Equation 4.5 and Equation 4.6 require

\[
\frac{\sqrt{8\tau T}}{2} \ll q_t \quad (4.20)
\]

where this restriction could be severer for small \( \tau \), because the charge variance still plays a significant role in this regime. Second, the accumulated measurement noise contribution
in the state variable Equation 4.11 is tolerable if
\[ \sqrt{\frac{T}{8\tau}} \ll q_t. \quad (4.21) \]

To sum up, the preservation of memory effects up to time \( T \) for a circuit with a signal \( q_t \) restricts the projection frequency from both sides,
\[ \frac{T}{8q^2_t} \ll \tau \ll \frac{q^2_t}{8T} \quad (4.22) \]

**Conditions on \( \tau \) to provide a localised trajectory**

The authors of [13] establish conditions on the measurement strength for continuous variable systems providing classicalisation in the form a well-localised trajectories. In their context, they are especially concerned with the emergence of chaotic dynamics from an underlying quantum mechanical system. A well-localised trajectory has one primary requirement, namely, its spread in phase space always remains small in comparison with the scale of non-linearities of the forces acting on the system. In this case, the Ehrenfest equations for the first moments coincide with the classical equations of motion. A continuous measurement with sufficient strength meets this requirement, since it constantly suppresses the spread of the variances. On the other hand, it induced random kicks in the first moments. Their amplitudes, accordingly the measurement strength, have to be sufficiently small to still allow for a smooth trajectory.

They discuss the case of monitoring position, whereas the charge measurement used above corresponds to a momentum monitoring. Still, their arguments should also hold for the present case, because the unitary contribution in the dynamics transform noise in one conjugate variable into noise in the other one.

The first condition requires that the spatial scale of the variation of the forces acting on the system is large compared to the spread of the wave packet. Thus, every part of the wave packet evolves identically. In the linear case of the LC oscillator considered above, this is always fulfilled. However, this condition is important for the quantum memristor, if the circuit involved non-linear elements.

The second condition provides a quasi-smooth trajectory, and reads [13, p. 6]
\[ \frac{2|\partial_x F|}{s} \ll \hbar k \ll \frac{|\partial_x F|s}{4}, \quad (4.23) \]
where \( F = -\partial_x U(x) \), \( k \) is the position measurement strength, and \( s \) is the typical action of the system in units of \( \hbar \).
The origin of both inequalities for the variable pair \( q, \varphi \) comes from the requirement,

\[
\Delta q \Delta \varphi \ll \delta q \cdot \delta \varphi = \hbar s.
\]

In one characteristic period of the system the changes \( \Delta q \) and \( \Delta \varphi \) due to the noise should be small compared to the changes due to the dynamics \( \delta q \) and \( \delta \varphi \). While the right hand side corresponds to the action of the system, the left hand side has two interpretations, corresponding to two two effects of the measurement strength \( k \). First, the variances due to the phase space distribution, which are suppressed with \( \frac{1}{k} \). Second, the fluctuations due to the back action, which grow with \( k \).

For a charge monitored LC circuit in unitless frequencies, the equivalent of Equation 4.23 is

\[
\frac{2}{s} \ll \tau \ll 4s. \quad (4.24)
\]

By identifying the typical unitless action of the oscillator with the average photon number \( \langle \mathcal{E} \rangle / \hbar \approx q_t^2 \), this inequality has the same scaling with respect to \( q_t \) as the restriction in Equation 4.22 for the memory effects.

Localised trajectories necessary, but not sufficient for a long term hysteresis

In Figure 4.5, we schematically present the restrictions on \( \tau \) and indicate the regime tested in the case study. Due to the high initial charge, it presents a situation far in the classical regime, where a big range of \( \tau \) fulfils the localisation condition. Equally, the memory condition is fulfilled for a big range of \( \tau \), as long as the time \( T \) is small, but it shrinks drastically when \( T \) increases. Correspondingly, the hysteresis for all tested \( \tau \) survived during the first oscillation, but afterwards collapsed for the non-optimal \( \tau \).

The second regime indicated in Figure 4.5 tests smaller initial loading of the LC circuit for the optimal choice of \( \tau \). Despite this choice and the fact that still the localisation condition is fulfilled, the hysteresis collapses faster for smaller loads. The noise growth in time remains, but the signal is reduced. In Figure 4.11, we show initial flux loads from 2 to 10.

Concluding, a memory effect in the unconditioned evolution is present until at least a time \( T \approx 1 \), only if the constraints on the projection frequency leading to localised trajectories in the conditioned evolutions are fulfilled. Still, the measurement noise in the feedback inevitably shrinks the memory at a later time, even if the conditioned trajectories remain localised. Hence, one might argue that the memory regime of our model is contained in the classical regime according to [13].
Chapter 4. *Memristor in a quantum LC circuit*

Figure 4.5: The classicalisation condition on $\tau$ and its dependence on the initial load, i.e. initial population, of the circuit. If the circuit is very populated the condition are fulfilled for a wide range of projection frequencies. The case study of this chapter has been done in this regime (arrow 1) and has proven the appearance of memory effects. Keeping the value of $\tau$ fixed and decreasing the load (arrow 2) the memory becomes corrupted faster indicating that the limits of the classical regime also restrict the presence of memory.

From this, two important questions emerge. On the one hand, is this conclusion true in general, if the memory is induced by an actuator using a classical record of the past? Regarding any basis e.g. a classical variable like position, quantum systems always offer multiple histories. If the tempting hypothesis, a classical state variable can only handle one history reliably, is true, then memory induced by a feedback scheme only works for rather classical systems. On the other hand, being aware of this tendency, how can one take advantage of memory effects in quantum technologies? Are there tasks, which can simultaneously profit from the memory feature and the quantum gain despite the small remaining coherences?
Chapter 4. Memristor in a quantum LC circuit

Figure 4.6: Slow measurement case, $\tau = 0.005$, of the evolution of an initially coherent state. The state variable update is dominated by noise. However, the back action of the measurement on the first moments is almost vanishing and the variances evolve towards their thermal values. Parameters are: $\gamma_0 = 0.1, \epsilon = 0.5, \lambda = 10, \tau = 0.05, \nu = 0.1, \varphi(0) = 20, q(0) = 0, V_\varphi = 0.5, V_q = 0.5, C_{\varphi,q} = 0, \mu(0) = 0$ and the trajectory has been simulated by a Euler algorithm with $dt = 10^{-3}$. 
Figure 4.7: Fast measurement case, $\tau = 4$, of the evolution of an initially coherent state. The state variable evolves nearly noise free as the high projection frequency suppresses the noise in the measurement signal. But this fast measurement introduces noise into the first moments. Parameters are: $\gamma_0 = 0.1, \epsilon = 0.5, \lambda = 10, \tau = 0.05, \nu = 0.1, \varphi(0) = 20, q(0) = 0, V_\varphi = 0.5, V_q = 0.5, C_{\varphi,q} = 0, \mu(0) = 0$ and the trajectory has been simulated by a Euler algorithm with $dt = 10^{-3}$. 
Figure 4.8: Illustration of the diffusion process in the damping rate due to the measurement noise. The black lines with different line structures correspond to four randomly chosen trajectories. Clearly visible is the contrasting character of their dephasing in the slow and fast measurement case, i.e. (A) and (C). The near optimal value $\tau \approx 0.2$ is shown in (B). All the trajectories follow up to fluctuations very neatly the damping rate evolution in the classical circuit (red), which in this case is almost indistinguishable from the ensemble mean (blue).
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Figure 4.9: The collection of figures summarises the mean evolution of the first moments of charge $\langle q \rangle$ and flux $\langle \varphi \rangle$ (top) and the departure from factorisation $\delta q = \langle \gamma(\mu) q \rangle - \langle \gamma(\mu) \rangle \langle q \rangle$ for the damping rate-charge correlation and the damping rate-flux correlation (bottom). The red line represents the classical noise free evolution. The ensemble size was $N = 3000$ and the estimation errors, not including numerical errors, are depicted by the transparent broadening of the curves.
Figure 4.10: Zoom into the region around the origin of Figure 4.4a revealing the non-pinched hysteresis due to the deviation $\delta_q(t)$ from factorisation of $\langle \langle \gamma(\mu)q \rangle \rangle$. Only the first one and a half oscillations are shown for clarity. The black line corresponds to the classical I-V curve. The red bars denote the ensemble estimation error, not including numerical errors due to the Euler Algorithm.
Figure 4.11: Testing the regime 2 in Figure 4.5. The initial flux load is reduced. Note that to get the same classical hysteresis the memory frequency has to be increased correspondingly i.e. here it is fixed as $\nu \varphi(0) = 2$. The hysteresis collapses faster as the signal is reduced while the noise stays the same. Note that for very small loads, i.e. a very low photon number in the LC oscillator, the correlation $\langle \gamma(\mu | q) \rangle$ and the deviation $\delta_q$ become comparable. This leads to the unexpected behaviour of the last two graphs, where the charge in the first oscillation exceeds the classical behaviour and even an additional crossing is observed. Estimation errors due to the finite ensemble size ($N=3000$) are omitted for visibility.
Chapter 5

Conclusions & Outlook

Quantum memristor dynamics

In this thesis, we set out to find a formalism of classical memristors in circuits with quantised electrical signals to test how their memory behaves in a quantum realm. We define these quantum systems with a history-dependent damping as quantum memristors. Our main result is a proposal of a quantum memristor by a measurement-based feedback approach. For this system, we established a phenomenological master equation, combining a Caldeira-Leggett type equation and the stochastic master equation that describes the quantum dynamics induced by continuous measurement.

To our knowledge, this has been the first trial to describe the dynamics of a quantum memristor. Hence, in this thesis we proposed a way to close a gap, left behind in the work of Di Ventra and Pershin [9], about the quantisation of memory elements in circuits. However, at variance with their work which included the quantisation of the state variable of the memory element, we treat the state variable classically. Still, a quantum memristor modelled by our feedback approach shows a new behaviour due to the different quality of quantum measurements.

We examined the memory in the quantum memristor consisting of a quantum LC circuit coupled to a memristor. By a numerical treatment of the evolution of Gaussian states, we proved that the memory in form of the hysteresis persists and that the non-Markovian character of the classical memristor is transferred to the quantum evolution. However, we found that the hysteresis is diminished and identified noise in the measurement signal and measurement back action as the origin of this reduction. The hysteresis recovers its classical shape, if the circuit is sufficiently populated and the measurement strength is chosen to simultaneously keep the two noise sources under control.
Finally, we discussed whether this case of high circuit population and a carefully chosen measurement strength constitutes a classical regime of our model. We concluded that it provides well-localised phase space trajectories, which are a footprint of classical systems.

Open questions

The hypothesis suggested by our findings, that a classical state variable can only handle classical histories in the form of well-localised trajectories, needs further testing. It would be an informative task to apply the quantum memristor master equation to a circuit involving non-linear elements. This would allow a comparison with the quantised RSJ-model of a Josephson junction by Yurke and Denker [10].

An important further step is to realise the microscopic derivation of the quantum memristor master equation sketched at the end of chapter 3. This could not only shine light on the validity of the form our heuristic master equation, it would also allow to have a more detailed view on the relation between non-Markovian feedback and quantum non-Markovianity. Since quantum non-Markovianity is discussed as beneficial in some quantum technology tasks [8, p. 45] and feedback provides a very controlled operation, it would be rewarding to understand this connection.

Moreover, it is interesting to think of possibilities to engineer a quantum memristor in the lab. One promising approach could be to use the controllable coupling between superconducting resonators and transmission lines via SQUIDs [53, 54]. Regarding the resonator as the system and the transmission line as the environment of harmonic oscillators, this provides the tuneable resistor in feedback model of the memristor. One could also build on a similar model of tuneable electromagnetic environments for qubits [55].

Further developments

A very recent paper by the group of Devoret experimentally investigates the influence of the memristive contribution present in the Josephson junction to the relaxation of a qubit [56]. They regard this term as a possibility to tune the dissipation of the qubit,
but do not consider it with respect to memory effects. The memristor is a role model of beneficial memory effects, proving the value to search for memory in existing quantum technologies and for ways to turn this additional feature into applications.

The memristor appears in many classical circuit models of biological systems. Di Ventra and others have proposed learning circuits using memory elements to mimic the adaption schemes of simple organisms [57]. Along the idea of quantum biomimetics [58], this poses the question, whether learning also works in the quantised version of this adaptive circuits. Our model of quantum memristors could be used to address this question. Another exciting starting point for further research in this direction is the similarity of the memristor and the synaptic connections in the brain. Networks in which quantum information is exchanged between distant processing units could be extended via learning links that adapt to the states that have passed through them.

Regarding a link in such a network as a quantum channel, quantum memristor connect to the concept of quantum channels with memory [59]. Along these lines, it is especially interesting to reconsider the feature of functional polymorphism for quantum logic gates. Can a quantum gate be enriched by an internal degree that allows a history-dependent response in form of a different mapping of input states? In other words, can a quantum memristor be a building block for a new concept of quantum information processing?

The joint venture of the classical memristor and the field of quantum technologies has various facets. This thesis has shed light on one of this facets and we hope that it stimulates further research exploring the rich physics behind memory, quantum mechanics and dissipation.
Appendix A

Stochastic differential equations

In this appendix, some important notions about stochastic differential equations (SDE) from the book on quantum feedback by Wiseman and Milburn [12, Appendix B] are summarised. Further helpful sources are the book on stochastic methods by Gardiner [23] and the book of Jacobs on quantum measurement containing a full calculation for a quantum feedback process [28, pp.494-497].

The difference between an implicit and an explicit SDE is best illustrated by the difference of the Stratonovich and the Ito form.

The Stratonovich form of a SDE for a random variable $X$ reads

$$\dot{X}(t) = \alpha(X) + \beta(X)\zeta(t). \quad (A.1)$$

The first part gives the deterministic update, whereas the second one is stochastic, driven by Gaussian white noise $\zeta(t)$. White noise defines an idealised noise process, whose mean vanishes and whose time correlation function is singular,

$$\langle\langle \zeta(t) \rangle\rangle = 0, \quad (A.2)$$

$$\langle\langle \zeta(t)\zeta(t') \rangle\rangle = \delta(t-t'). \quad (A.3)$$

Gaussian white noise further defines the Wiener increment,

$$dW(t) = \zeta(t)dt. \quad (A.4)$$

Hence, the Wiener increment is a random increment with the peculiarity that its square is a deterministic variable which is equal to the time increment,

$$\langle\langle dW(t) \rangle\rangle = 0, \langle\langle dW(t)^2 \rangle\rangle = dW(t)^2 = dt. \quad (A.5)$$
The explicit increment of $X$ is defined by

$$dX(t) = X(t + dt) - X(t).$$  \hfill (A.6)

The Ito form of the above SDE provides this explicit increment and reads

$$dX(t) = \alpha(X(t))dt + \frac{1}{2} \beta'(X(t))\beta(X(t))dt + \beta(X(t))dW(t).$$  \hfill (A.7)

The additional deterministic term $\frac{1}{2} \beta'(X(t))\beta(X(t))dt$ is the crucial difference between the implicit Stratonovich and the explicit Ito form. The origin of this difference is the singular nature of the Gaussian white noise, which does not allow to cut the Taylor expansion after the first order. Higher order noise terms can be comparable to $dt$, like, in this case, the square of the Wiener increment. A useful property of the Ito form is that the solution $X(t)$ and the Wiener increment $dW(t)$ are independent, i.e. $\langle \langle X(t)dW(t) \rangle \rangle = \langle \langle X(t) \rangle \rangle \langle \langle dW(t) \rangle \rangle = 0$.

The transformation to an explicit increment is important for a numerical treatment. If the noise term in the implicit equation is of a different form than Gaussian white noise, there is a general formula for this transformation based on the Taylor expansion, see [12, App. B, Eq. (B.64)].

For Gaussian noise, the simplest numerical simulation of a SDE is the explicit Euler algorithm based on a discrete version of the Ito form,

$$X(t_{j+1}) = X(t_j) + \alpha(X(t_j))\Delta t + \frac{1}{2} \beta'(X(t_j))\beta(X(t_j))\Delta t + \beta(X(t_j))G_j\sqrt{\Delta t}. \hfill (A.8)$$

The time increment $\Delta t$ has to be chosen sufficiently small and the $G_j$ are independently drawn from a Gaussian distribution with mean 0 and variance 1. A comprehensive evaluation of this algorithm and a review of more elaborate numerical treatments of SDEs is given in [23, ch.10].

In this thesis, the numerical simulations are based on the explicit Euler algorithm, because, in the case study of Chapter 4 on the quantum memristor master equation, the main interest is a qualitative understanding. Although Mathematica also uses the Euler algorithm as their default method, here an own implementation in Matlab is used and the code is gladly made available upon request\textsuperscript{1}. As a small test of the code, Figure A.1 shows a comparison of the results for a simulation of the quantum memristor master equation between Matlab and Mathematica. The time dependence of the mean damping rate is displayed, see Chapter 4. Qualitatively, the behaviour is the same. However, Mathematica does not easily allow to handle large ensemble sizes. Our code creates the

\textsuperscript{1}Contact: pfeifferpaul90@gmail.com
Figure A.1: Test of the Matlab code via a comparison of the evolution of the mean value damping rate with a solution obtained in Mathematica. The different lines depict different time increments $dt$. The ensemble size are: 3000 (Matlab), 300 (Mathematica).

trajectories one by one and stores them on the hard disc, so that it might access much larger ensemble sizes. This is the reason for the quantitative difference for later times for which the variance of the damping rate has grown. The insufficient ensemble size of the Mathematica simulation is also visible in the remaining fluctuations.
Appendix B

Gaussian states

A thorough introduction to Gaussian states is given for example in [60]. Here two results that are used in the thesis are provided. Let $\rho$ be a Gaussian state and $\varphi, q$ denote the conjugate variable pair. The second order cumulants are

$$V_{\varphi} = \langle \varphi^2 \rangle - \langle \varphi \rangle^2,$$  \hspace{1cm} (B.1)

$$V_q = \langle q^2 \rangle - \langle q \rangle^2,$$  \hspace{1cm} (B.2)

$$C_{\varphi,q} = \frac{1}{2} \langle \{\varphi, q\} \rangle - \langle \varphi \rangle \langle q \rangle.$$  \hspace{1cm} (B.3)

Together with the first moments, $\langle \varphi \rangle$ and $\langle q \rangle$, the variances contain all the information necessary to parametrise the Gaussian state. Hence, higher order moments can be expressed in terms of these 5 parameters. Here, only the ones necessary to reproduce the calculation done in Section 2.2.2.3 are presented,

$$\langle q^3 \rangle = \langle q \rangle^3 + \langle q \rangle V_q,$$  \hspace{1cm} (B.4)

$$\frac{1}{2} \langle \{q, \varphi^2\} \rangle = \langle q \rangle \langle \varphi \rangle^2 + 2 \langle \varphi \rangle C_{\varphi,q} + \langle q \rangle V_{\varphi},$$  \hspace{1cm} (B.5)

$$\frac{1}{2} \langle \{q, q, \varphi\} \rangle = \langle q \rangle \langle \varphi \rangle^2 + 2 \langle q \rangle C_{\varphi,q} + \langle \varphi \rangle V_q.$$  \hspace{1cm} (B.6)

Gaussian states allow many important quantities to be calculated analytically. For example, the purity $\Pi = \text{Tr} \rho^2$ reads

$$\Pi = \frac{1}{2 \sqrt{\text{det} \sigma}},$$  \hspace{1cm} (B.7)

with the covariance matrix $\sigma = (V_{ij} = \frac{1}{2} \langle \{i, j\} \rangle - \langle i \rangle \langle j \rangle)$ [60, eq. (2.5)].
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